The Economics of Space 433: Lectures 11 and 12

Solving for Spatial Equilibria

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Geography and Economic Allocations

- We will now take into account the assymetric geography and attempt to understand the distribution of economic activity across space
  - We need a proper theory that takes into account the geography of locations
  - Much of what follows is based on the theory laid out on the Spatial Primer in the class website, co-written with Treb Allen.
  - Hereby, I will present a simplified version and some key results

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- Our framework is an environment that constitutes a synthesis of the traditional approach of urban and regional economics and the modern approach of modeling frictions in space

Roadmap

- **The Concept of General Equilibrium**
- General Equilibrium Connections: Market Access Spillovers
- General Equilibrium: Solving for Wages and Labor
- Agglomeration Forces and the Possibility of Multiple Equilibria

Setup

- Set of locations $S = \{1, 2\}$ (could be arbitrarily many)
  - Each location produces a differentiated commodity. Convention: origin denoted by i, destination by j
    - Same notation for locations and goods
  - Population of location $j = 1, 2$, denoted by L_j.
  - Total population
$$\bar{L} = \sum_j L_j \tag{1}$$
  - Topography economy: Productivities, amenities, trade costs
  - Firms and consumers maximize. Take prices as given

The Concept of General Equilibrium

- So far we have taken P_i, Π_i as given and solve w_i, L_i as a function of those.
- To solve for P_i, Π_i, we need to solve for the joint interactions of firms and consumers across markets
  - We indeed considered the interactions of firms and consumers within a market
  - Given market access
- To solve for what we call ''general equilibrium'' we need to ALSO consider the market access interactions across locations
  - Ultimately, the solution of the general equilibrium entails solving w_i, L_i

The Equations of General Equilibrium

- The first two are rather simple ones: Adding up constraint on labor and welfare equalization

$$\bar{L} = \sum_i L_i \ , \ \bar{W} = W_i$$

- To ''close'' the model we need to discuss what happens to total output.
  - It is all labor income

$$\underbrace{w_i L_i = Y_i}_{\text{income of } i} = \underbrace{\sum_j \left(\frac{A_i}{w_i}\right)^{\sigma-1} \tau_{ij}^{1-\sigma} \frac{w_j L_j}{P_j^{1-\sigma}}}_{\text{sales to all destinations from } i} = \sum_j \underbrace{\left(\frac{\frac{w_i \tau_{ij}}{A_i}}{P_j}\right)^{1-\sigma}}_{\text{share of } i} \underbrace{w_j L_j}_{\text{income of } j}$$

  - Synonymous (in this setting) to a labor market clearing condition

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  - Synonymous (in this setting) to a labor market clearing condition
  - We can intuitively analyze what forces determine the determination of the general equilibrium using the concept of market access pioneered by Anderson van Wincoop '03 and Redding Venables '04
  - Note: We will go back and analyze solution of market access using $\bar{W} = W_i$

Market Access

- Using $A_i = \bar{A}_i L_i^{\alpha}$, we can derive labor demand (given Π_i)

$$w_i L_i = \left(\frac{\bar{A}_i}{w_i}\right)^{\sigma-1} L_i^{\alpha(\sigma-1)} \underbrace{\sum_j \tau_{ij}^{1-\sigma} \frac{w_j L_j}{P_j^{1-\sigma}}}_{\text{producer market access}}$$

Thus, labor demand (set $L_i = L_i^D$) is

$$w_i^{\sigma} = \bar{A}_i^{\sigma-1} \left(L_i^D\right)^{\alpha(\sigma-1)-1} \Pi_i \tag{2}$$

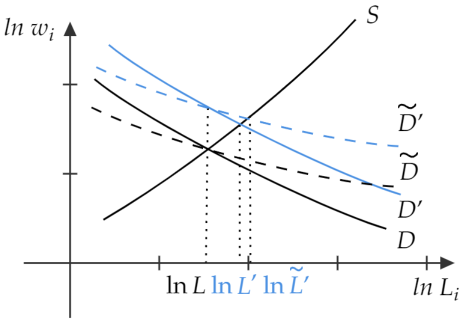
- Now note that supply is given by $\bar{W} = \frac{w_i}{P_i} \times \bar{u}_i \left(L_i^S\right)^{-\beta}$. Thus, (given P_i)

$$w_i = \left(L_i^S\right)^{\beta} \bar{W} (\bar{u}_i)^{-1} P_i \tag{3}$$

Slope of Demand

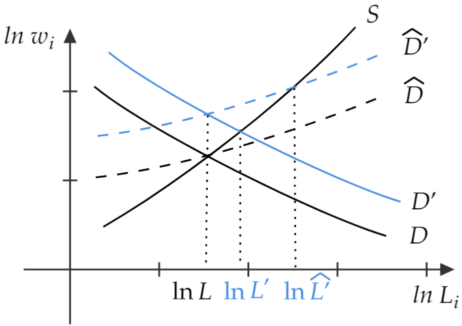
- What are the implications of a demand that is more elastic or even upward sloping?
- Allocations are more sensitive to a change in the "shifters", $\bar{u}_i, \bar{A}_i, P_i$
- Let us examine what happens for a change in $\bar{u}_i, \bar{A}_i$ in different scenarios

Slope of Demand



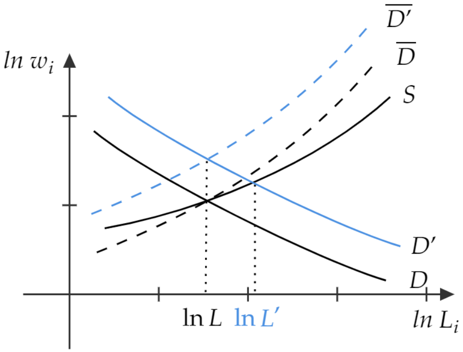
- When $\bar{A}_i$ increases ($D \to D', \tilde{D} \to \tilde{D}'$), labor responses more with more elastic demand ($L' < \tilde{L}'$).

Slope of Demand



- When $\bar{A}_i$ increases ($D \to D', \hat{D} \to \hat{D}'$), labor responses even more with upward sloping demand ($L' < \tilde{L}' < \hat{L}'$)

Slope of Demand



- If the demand curve $\bar{D}$ is steeper than the supply curve, the economy is unstable.

Roadmap

- The Concept of General Equilibrium
- **General Equilibrium Connections: Market Access Spillovers**
- General Equilibrium Characterization: Solving for Wages and Labor
- Agglomeration Forces and the Possibility of Multiple Equilibria

Slope of Demand

- How does economic activity propagate across space?
- We need to look at how market access changes are linked across space
  - When market access in a location changes, what happens to other locations
  - These are determined through a multi-location equation solution
- We need one more equation
  - We derive this using the price index definition

Market Access and Demand Spillovers

- We have that

$$P_j^{1-\sigma} \equiv \sum_{i'} p_{i'j}^{1-\sigma} = \sum_{i'} \tau_{i'j}^{1-\sigma} \frac{\left(w_{i'}\right)^{1-\sigma}}{\bar{A}_{i'}^{1-\sigma} L_{i'}^{\alpha(1-\sigma)}}$$

- Replacing the labor supply (equation 3)

$$P_j^{1-\sigma} = \sum_{i'} \tau_{i'j}^{1-\sigma} \frac{\left(\bar{W} L_{i'}^{\beta} \frac{P_{i'}}{\bar{u}_{i'}}\right)^{1-\sigma}}{\bar{A}_{i'}^{1-\sigma} L_{i'}^{\alpha(1-\sigma)}}$$

$$\bar{W}^{\sigma-1} P_j^{1-\sigma} = \sum_{i'} \bar{u}_{i'}^{\sigma-1} \tau_{i'j}^{1-\sigma} \bar{A}_{i'}^{\sigma-1} L_{i'}^{(\alpha-\beta)(\sigma-1)} \left(P_{i'}\right)^{1-\sigma} \tag{4}$$

  - Consider first the case $\alpha = \beta$. Take two countries, symmetric $\bar{u}_i = \bar{A}_i = 1$

$$\bar{W}^{\sigma-1} P_1^{1-\sigma} = P_1^{1-\sigma} + \tau^{1-\sigma} \left(P_2\right)^{1-\sigma}$$

  i.e. changes in $P_2^{1-\sigma}$ are passed through (incompletely) on $P_1^{1-\sigma}$

    - A feedback effect through space.

Market Access and Demand Spillovers

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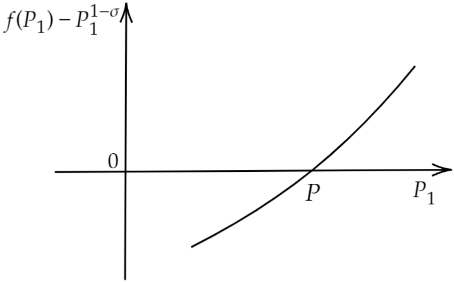
$$\bar{W}^{\sigma-1} P_j^{1-\sigma} = \sum_{i'} \bar{u}_{i'}^{\sigma-1} \tau_{i'j}^{1-\sigma} \bar{A}_{i'}^{\sigma-1} L_{i'}^{(\alpha-\beta)(\sigma-1)} \left(P_{i'}\right)^{1-\sigma} \tag{5}$$

  - Note: presence of $L_{i'}^{(\alpha-\beta)(\sigma-1)}$ magnifies own market access responses if $\alpha > \beta$

$$\bar{W}^{\sigma-1} P_1^{1-\sigma} = L_1^{(\alpha-\beta)(\sigma-1)} P_1^{1-\sigma} + L_2^{(\alpha-\beta)(\sigma-1)} \tau^{1-\sigma} \left(P_2\right)^{1-\sigma}$$

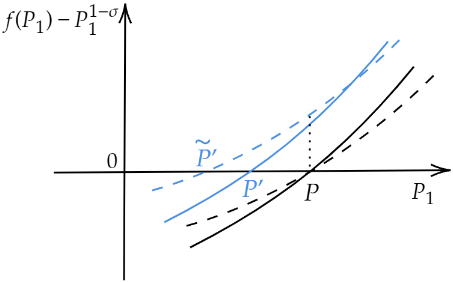
    - Location with higher L_i gets a bigger kick

Market Access and Demand Spillovers



- Define $f(P_1) = (\bar{W})^{1-\sigma} \sum_{i'=1,2} \bar{u}_{i'}^{\sigma-1} \tau_{i'1}^{1-\sigma} \bar{A}_{i'}^{\sigma-1} L_{i'}^{(\alpha-\beta)(\sigma-1)} (P_{i'})^{1-\sigma}$. Set $P_2 = 1$
- Equilibrium is determined by the intersection between $f(P_1) - P_1^{1-\sigma}$ and 0.

Market Access Demand Spillovers



- If $\bar{A}_1$ increases, an equilibrium price P_1 decreases from P to P'.
- If $\alpha - \beta$ is larger (dotted line), the price responds more ($\tilde{P}' < P' < P$)

Roadmap

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- **General Equilibrium: Solving for Wages and Labor**
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Demand and Supply

- Use the demand equation (2):

$$w_i^\sigma = \bar{A}_i^{\sigma-1} \left(L_i^D\right)^{\alpha(\sigma-1)-1} \Pi_i \iff w_i = \bar{A}_i^{\frac{\sigma-1}{\sigma}} \left(L_i^D\right)^{\frac{\alpha(\sigma-1)-1}{\sigma}} \Pi_i^{\frac{1}{\sigma}}$$

- Consider supply, equation (3),
  - Set them equal with $L_i^D = L_i^S = L_i$,

$$\bar{A}_i^{\frac{\sigma-1}{\sigma}} (L_i)^{\frac{\alpha(\sigma-1)-1}{\sigma}} \Pi_i^{\frac{1}{\sigma}} = (L_i)^\beta \bar{W}(\bar{u}_i)^{-1} P_i \iff$$

$$\bar{A}_i^{\frac{\sigma-1}{\sigma}} (L_i)^{\frac{\alpha(\sigma-1)-1}{\sigma}} \Pi_i^{\frac{1}{\sigma}} = (L_i)^\beta \bar{W}(\bar{u}_i)^{-1} P_i$$

- Assume, symmetry of trade costs, $\tau_{ij} = \tau_{ji}$, which implies $\Pi_i = P_i^{1-\sigma}$ (proof ommited)
  - We obtain

$$\bar{W}^\sigma L_i^{1+\beta\sigma-\alpha(\sigma-1)} = \bar{u}_i^\sigma \bar{A}_i^{\sigma-1} P_i^{1-2\sigma} \tag{6}$$

Demand and Supply

- Let us obtain some intuition from this equation

$$L_i^{1+\beta\sigma-\alpha(\sigma-1)} = \frac{\bar{u}_i^{\sigma} \bar{A}_i^{\sigma-1} P_i^{1-2\sigma}}{\bar{W}^{\sigma}}$$

- Assume substitutes, $\sigma > 1$, as always
  - Ceteris paribus, increases in innate characteristics $\bar{u}_i, \bar{A}_i$ increase L_i
  - Same for lower P_i, higher market access

- How does a foreign shock propagate?
  - It depends on how connected the markets are
  - Increase in $\bar{A}_j \implies P_i \downarrow$ but also $\bar{W} \uparrow$.
  - If j is close to i then P_i drop will dominate and L_i will increase
  - Vice versa if far away

The General Equilibrium System

- The final solution of the system is given by (4), (6), and the adding up constraint $\bar{L} = \sum_i L_i$
- We have the system

$$\bar{W}^{\sigma-1} P_i^{1-\sigma} = \sum_{i'} \bar{u}_{i'}^{\sigma-1} \tau_{i'i}^{1-\sigma} \bar{A}_{i'}^{\sigma-1} L_{i'}^{(\alpha-\beta)(\sigma-1)} \left(P_{i'}\right)^{1-\sigma} \tag{7}$$

$$\bar{W}^{\sigma} L_i^{1+\beta\sigma-\alpha(\sigma-1)} = \bar{u}_i^{\sigma} \bar{A}_i^{\sigma-1} P_i^{1-2\sigma} \tag{8}$$

that solves for P_i, L_i and $\bar{W}$

- With symmetry and $\Pi_i = P_i^{1-\sigma}$ we can solve for w_i using (2)

A Simple Solution. Many Locations Placed on a Line

- Now we will attempt an analytical characterization of this model with locations placed on a line
  - Set $u_i, A_i = 1$ (symmetry, no spillovers $\alpha = \beta = 0$)
  - Restrict geography of trade costs $\tau_{ij} = e^{\bar{t}|i-j|}$. $\bar{t}$ is elasticity of trade cost to distance

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- Population can be determined by a differential equation (in space)
  - In natural sciences, we solve for the energy of each point in the system
    - Energy is determined by whether a point is well placed to other high-energy points

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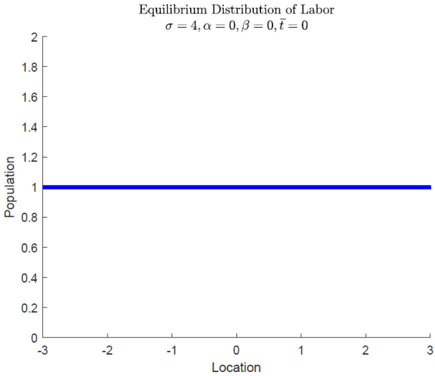
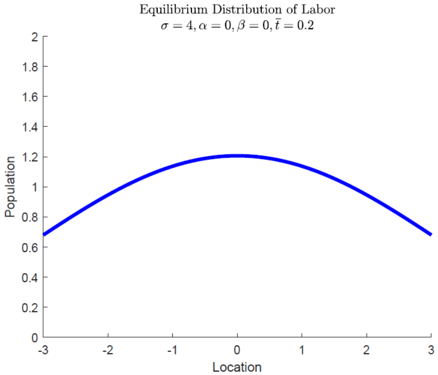
- Population can be determined by a differential equation (in space)
  - In natural sciences, we solve for the energy of each point in the system
    - Energy is determined by whether a point is well placed to other high-energy points
  - Here, locations that are well-placed will attract more people
    - The spatial link is trade!

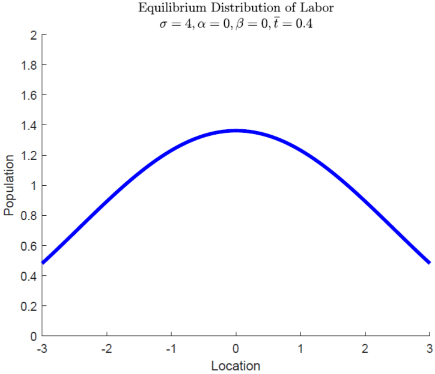
Solution on a Line with $\bar{t} = 0$



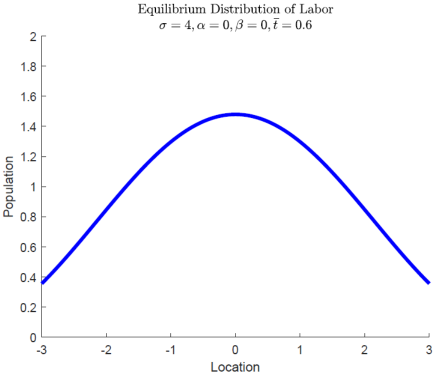
Solution on a Line with Small $\bar{t}$



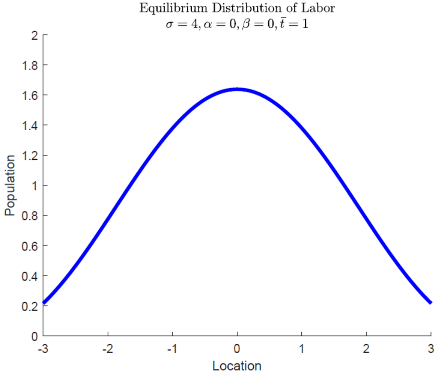
Solution on a Line with Larger $\bar{t}$



Solution on a Line with Large $\bar{t}$



Solution on a Line with Very Large $\bar{t}$



Building a Border (aka a Wall)

- L_i is the solution of a differential equation in space, same as the **pendulum** in time
  - Solution of this differential equation is a very familiar function

$$L(i) = (c_1 \cos(ki))^{\frac{2\sigma-1}{\sigma-1}}$$

Building a Border (aka a Wall)

- L_i is the solution of a differential equation in space, same as the **pendulum** in time
  - Solution of this differential equation is a very familiar function

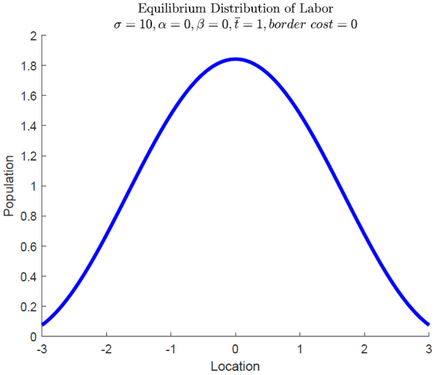
$$L(i) = (c_1 \cos(ki))^{\frac{2\sigma-1}{\sigma-1}}$$

- Now add a border in the middle (on top of trade cost)
  - The solution becomes

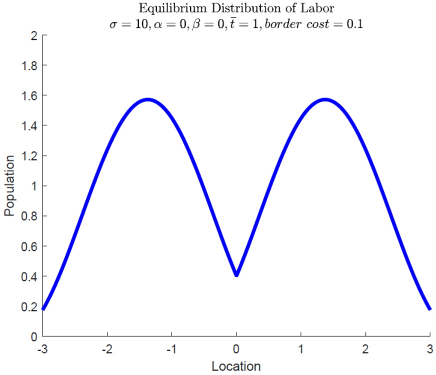
$$L(i) = (c_1 \cos(ki) + c_2 |\sin(ki)|)^{\frac{2\sigma-1}{\sigma-1}}$$

  - Same differential equation in space as the **spring** in time

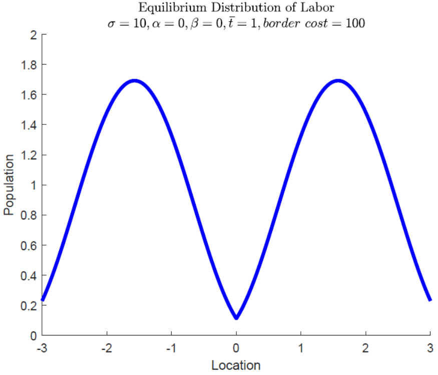
World without a Border



Building a Large Border



Building a Ridiculous Border



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Agglomeration and Dispersion Forces

- The implications of strong agglomeration$>>$dispersion forces could go beyond large L_i effects
- Going back to the example of a city like Detroit, it will lead to a reversal of fortunes
  - Oftentimes this is interpreted as a ''switch'' between equilibria when there are more than one
  - See Davis Weinstein '02

Agglomeration and Many Solutions: 2 Location Example

- Consider our spatial model (with welfare equalization). Lets consider the case of:
  - 2 symmetric locations in terms of geography: $\bar{A}_1 = \bar{A}_2$, $\bar{u}_1 = \bar{u}_2$
  - Trade costs symmetric, $\tau_{12} = \tau_{21} \equiv \tau > 1$
  - $\alpha > 0$ and $\beta = 0$
  - Normalize aggregate labor supply to 1, $\bar{L} = 1$

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  - Trade costs symmetric, $\tau_{12} = \tau_{21} \equiv \tau > 1$
  - $\alpha > 0$ and $\beta = 0$
  - Normalize aggregate labor supply to 1, $\bar{L} = 1$
- By combining our equilibrium equations (substitute 8 into 7) we can write system as a single equation

$$L_i^{\frac{\sigma-1}{2\sigma-1}\gamma_1} = \bar{W}^{1-\sigma} \sum_j \tau_{ij}^{1-\sigma} L_j^{\frac{\sigma-1}{2\sigma-1}\gamma_2} \tag{9}$$

where $\gamma_1 = \underbrace{\beta\sigma}_{\text{supply slope}} - \underbrace{(\alpha(\sigma-1)-1)}_{\text{labor demand slope}}$, $\gamma_2 = 1 + \alpha\sigma - (\sigma-1)\beta$

Agglomeration and Many Solutions: 2 Location Example

- Some algebra, plus the fact $L_1 + L_2 = \bar{L}$ =1, yields

$$\underbrace{L_1}_{\text{supply}} = \left[\bar{W}^{1-\sigma} \left(L_1^{\tilde{\sigma}\gamma_2} + \tau^{1-\sigma} \left(1 - L_1\right)^{\tilde{\sigma}\gamma_2} \right) \right]^{\frac{1}{\tilde{\sigma}\gamma_1}} \equiv \underbrace{f\left(L_1\right)}_{\text{demand}}$$

with

$$L_2 = \left[\bar{W}^{1-\sigma} \left(L_2^{\tilde{\sigma}\gamma_2} + \tau^{1-\sigma} L_1^{\tilde{\sigma}\gamma_2} \right) \right]^{\frac{1}{\tilde{\sigma}\gamma_1}} \iff \bar{W}^{1-\sigma} = \frac{\left(1 - L_1\right)^{\tilde{\sigma}\gamma_1}}{\left(\left(1 - L_1\right)^{\tilde{\sigma}\gamma_2} + \tau^{1-\sigma} L_1^{\tilde{\sigma}\gamma_2} \right)}$$

and $\tilde{\sigma} = \left(\sigma - 1\right) / \left(2\sigma - 1\right), \quad \gamma_1 = 1 - (\sigma - 1)\alpha, \quad \gamma_2 = 1 + \alpha\sigma$

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and $\tilde{\sigma} = (\sigma - 1)/(2\sigma - 1)$, $\quad \gamma_1 = 1 - (\sigma - 1)\alpha$, $\quad \gamma_2 = 1 + \alpha\sigma$

- $f\left(L_1\right)$ is the demand for labor in location 1. L_1 is the supply
  - We define $f\left(L_1\right) - L_1$ as the excess demand function (i.e. demand minus supply)

Excess Demand Function Analysis

- We can analyze equilibrium properties by analyzing the excess demand
  - If $f(L_1) - L_1 = 0$, then this is an equilibrium
  - If $f(L_1) > L_1$, there is an excess demand (and excess supply if $<$)

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- We can also introduce a new concept borrowing from General Equilibrium Theory: Stability
  - Start from equilibrium. Study excess demand of country 1 to see if stable
  - Assume you increase employment. If $f(L_1) - L_1 < 0$, equilibrium is stable
  - Assume you increase employment. If $f(L_1) - L_1 > 0$, equilibrium is unstable

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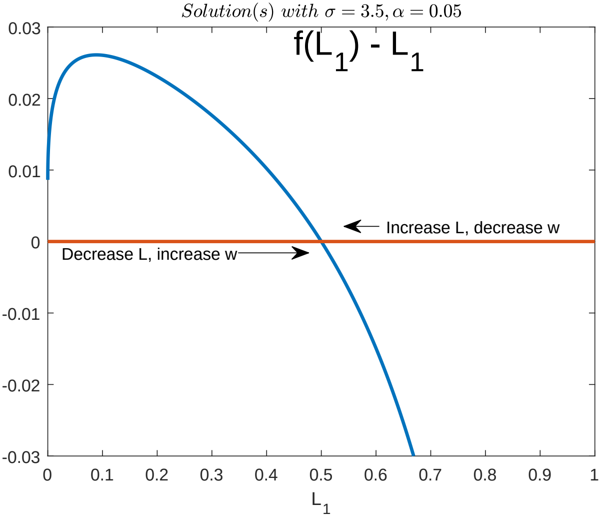
  - Assume you increase employment. If $f(L_1) - L_1 < 0$, equilibrium is stable
  - Assume you increase employment. If $f(L_1) - L_1 > 0$, equilibrium is unstable
- Intuition for unstable solution: if more employment means more demand for labor, wages increases and more workers flood in
  - Clearly unstable!

Agglomeration and Many Solutions: 2 Location Example



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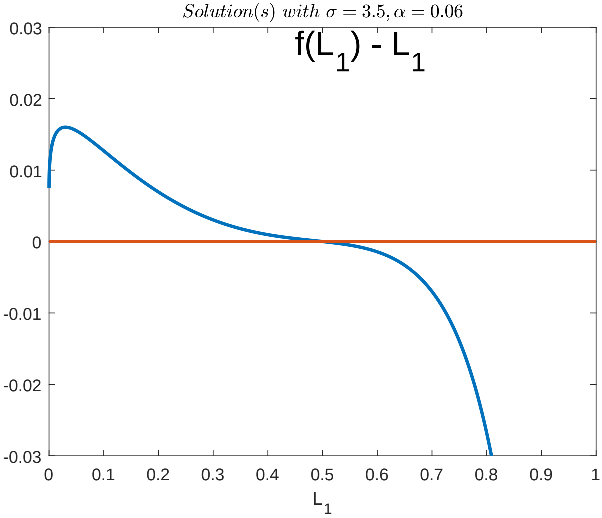

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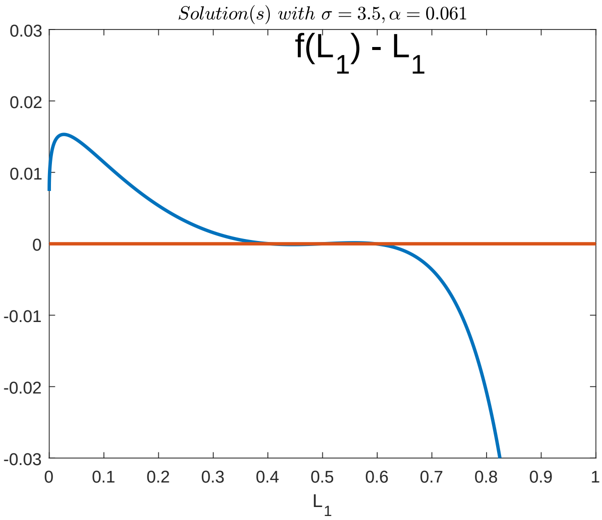

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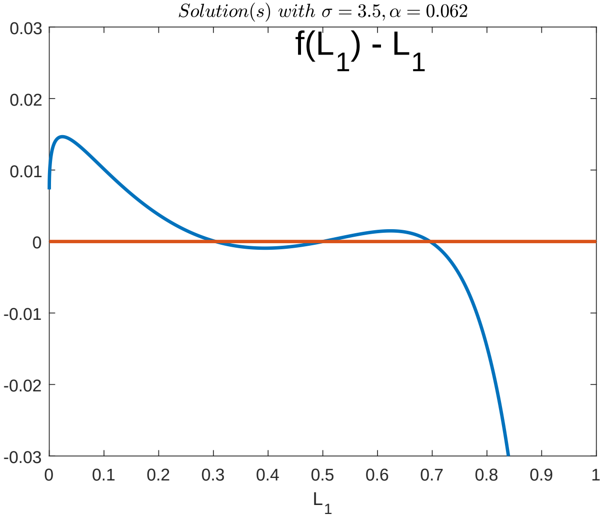
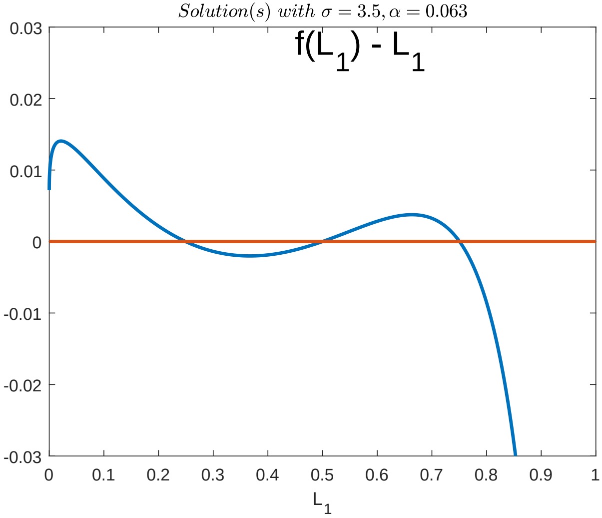
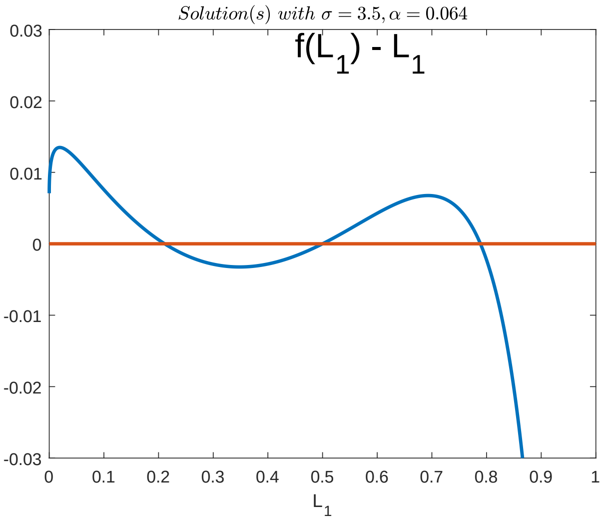


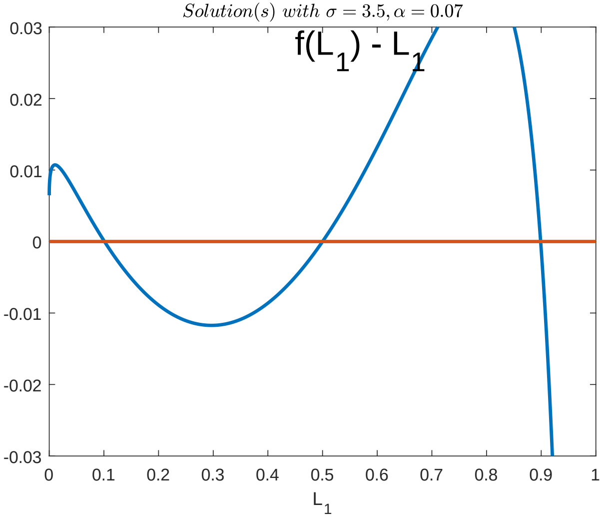
Agglomeration and Many Solutions: 2 Location Example



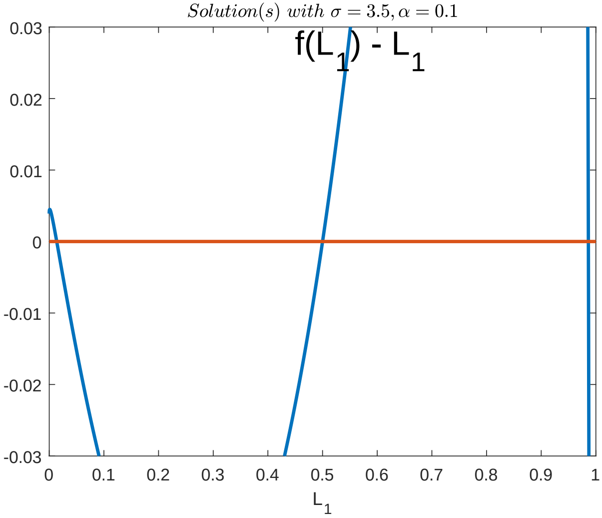
Agglomeration and Many Solutions: 2 Location Example



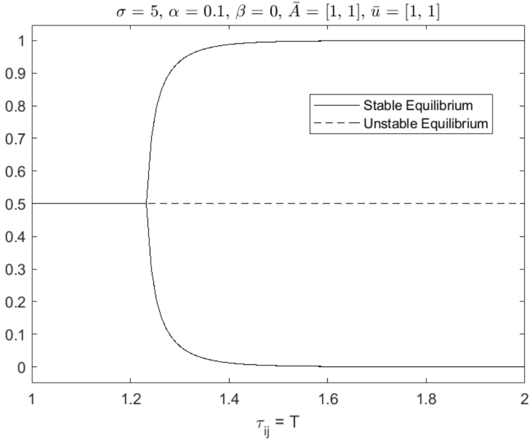
Agglomeration and Many Solutions: 2 Location Example



Agglomeration and Many Solutions: 2 Location Example



Agglomeration and Trade Costs: The Pitchfork



References

- Spatial Economics Primer, Allen and Arkolakis 2017. Mimeo.
- Gravity with Gravitas: A Solution to the Border Puzzle, Anderson and van Wincoop, 2003, American Economic Review, 93(1), pp. 170-192.
- Economic Geography and International Inequality, Redding and Venables, 2004, Journal of International Economics,62(1), pp. 53-82.
- Bones, Bombs, and Break Points: the Geography of Economic Activity, Davis and Weinstein, American Economic Review, 92(5), pp. 1269-1289.