The Economics of Space 433: Lectures 13 and 14

Solving for Spatial Equilibria

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Solving for Spatial Equilibria

- We have developed the theoretical apparatus to model the economic and geographic aspects of Space
- We will now proceed to show how to solve for spatial equilibria
- Intrinsic to the notion of solving spatial equilibria is the concept of a network
  - That is, if there are trade frictions

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Roadmap

- **Solving for the Spatial Equilibrium Without Network Effects**
- Improving Access and Geography: The Case of two Locations
- Effects of Improving Access on the Network of Locations
- Network Effects Analyzed
- Iterative Solution of the Equilibrium

Solving for the Spatial Equilibrium

- Consider our spatial model (with welfare equalization).
- Recall that the solution entails

$$w_i L_i = \sum_j \lambda_{ij} w_j L_j$$

  - Since $\lambda_{ij} = \left(\frac{w_i \tau_{ij}}{A_i}\right)^{1-\sigma} / P_j^{1-\sigma}$ recall that we can write

$$w_i L_i = \sum_j \frac{\left(\frac{w_i \tau_{ij}}{A_i}\right)^{1-\sigma}}{P_j^{1-\sigma}} w_j L_j \implies$$

$$w_i^{\sigma} L_i = A_i^{\sigma-1} \Pi_i$$

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- Let us refer to $\tilde{X}_i \equiv w_i^{\sigma} L_i$ as the location's "adjusted market potential"
  - The higher the productivity, A_i, and the producer market access, Π_i, the higher the market potential of a market (again assume $\sigma > 1$)

No Trade Costs

- There is a simple case, the one of no trade costs, $\tau_{ij} = 1$ for all i, j
  - Then
    $$w_i^{\sigma} L_i = A_i^{\sigma-1} \Pi$$
    so that market potential are proportional to productivity
  - This case roughly resembles the Rosen-Roback result ($w_i = A_i$)

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- What happens when $\tau_{ij} \neq 1$ for all i, j?
  - In that case ''network'' effects play a key role

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Network of Locations and Network Effects

- Let us assess network effects formally using the feasibility constraint

$$w_i L_i = \sum_j \frac{\left(w_i \tau_{ij} / \bar{A}_i\right)^{1-\sigma}}{P_j^{1-\sigma}} w_j L_j$$

Network of Locations and Network Effects

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$$w_i L_i = \sum_j \frac{\left(w_i \tau_{ij}/\bar{A}_i\right)^{1-\sigma}}{P_j^{1-\sigma}} w_j L_j$$

- Assuming no spillovers ($\alpha = \beta = 0$), $\bar{u}_i = 1$ for all i and replacing for welfare equalization

$$w_i L_i = \sum_j \frac{w_j^{1-\sigma}}{P_j^{1-\sigma}} \frac{\left(w_i \tau_{ij}/\bar{A}_i\right)^{1-\sigma}}{w_j^{1-\sigma}} w_j L_j = W^{1-\sigma} \sum_j \left(w_i \tau_{ij}/\bar{A}_i\right)^{1-\sigma} w_j^{\sigma} L_j \implies$$

$$w_i^{\sigma} L_i = W^{1-\sigma} \sum_j \left(\tau_{ij}/\bar{A}_i\right)^{1-\sigma} w_j^{\sigma} L_j$$

  - Let us refer to $\tilde{X}_i \equiv w_i^{\sigma} L_i$ as the locations ‘‘adjusted market potential’’

Effects of Improving Access

- To develop intuition let's consider the case of:
  - $\alpha = 0$ and $\beta = 0$ with 2 symmetric locations in terms of geography: $\bar{A}_1 = \bar{A}_2$, $\bar{u}_1 = \bar{u}_2$
  - Trade costs symmetric, $\tau_{12} = \tau_{21} \equiv \tau > 1$
  - Normalize aggregate labor supply to 1, $\bar{L} = 1$

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  - Trade costs symmetric, $\tau_{12} = \tau_{21} \equiv \tau > 1$
  - Normalize aggregate labor supply to 1, $\bar{L} = 1$

- We can write system as $L_i^{\frac{\sigma-1}{2\sigma-1}} = \bar{W}^{1-\sigma} \sum_j \tau_{ij}^{1-\sigma} L_j^{\frac{\sigma-1}{2\sigma-1}}$
  - (again, to be proven in your homework)

Effects of Improving Access: 2 Location Example

- Claim: We can write system as $L_i^{\frac{\sigma-1}{2\sigma-1}} = \bar{W}^{1-\sigma} \sum_j \tau_{ij}^{1-\sigma} L_j^{\frac{\sigma-1}{2\sigma-1}}$

- Since $\alpha = 0$ and there are only two countries

$$L_1^{\tilde{\sigma}} = W^{1-\sigma} \left(L_1^{\tilde{\sigma}} + \tau^{1-\sigma} L_2^{\tilde{\sigma}} \right) \iff W^{\sigma-1} = 1 + \tau^{1-\sigma} \left(\frac{1 - L_1}{L_1} \right)^{\tilde{\sigma}}$$

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- Assume symmetry

$$W^{\sigma-1} = 1 + \frac{1}{\tau^{\sigma-1}}$$

  - How does investment in τ affect welfare? Lower τ implies higher welfare
  - Without symmetry you also have population reallocations

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Network of Locations

- The theory we developed is effectively a network of locations linked through trade and population
- The intensity of the (spatial) links determines the degree of interaction
  - Recall: feasibility condition $w_i L_i = \sum_j \lambda_{ij} w_j L_j$.
  - We can now go back to our adjusted market potential equation

$$W^{\sigma-1}\tilde{X}_i = \sum_j \left(\tau_{ij}/\bar{A}_i\right)^{1-\sigma} \tilde{X}_j$$

- This equation has a linear format

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Total Differentiation of the System

- We will now start from our adjusted market potential equation

$$W^{\sigma-1}\tilde{X}_i = \sum_j \left(\tau_{ij}/\bar{A}_i\right)^{1-\sigma} \tilde{X}_j$$

- Next step, totally differentiate this expression. We exploit two properties of logarithms
  - $\ln(ab) = \ln a + \ln b$ and
  - $d\ln(a+b) = \frac{da}{a+b} + \frac{db}{a+b} = \frac{a}{a+b} d\ln a + \frac{b}{a+b} d\ln b$

- Consider two locations. Consider an investment in τ_{12}.

- Define $y_{ij} \equiv \frac{X_{ij}}{\sum_j X_{ij}} = \frac{(w_i\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j}{\sum_{j'}(w_i\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}} = \frac{(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j}{\sum_{j'}(\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}}$, the ''export share'' of location j within the total exports of location i

Total Differentiation of the System

- We have that we can write the system above as

$$(\sigma - 1)\, d \ln W + d \ln \tilde{X}_1 = (1 - \sigma)\, y_{12} d \ln \tau_{12} + \sum_j y_{1j} d \ln \tilde{X}_j$$

$$(\sigma - 1)\, d \ln W + d \ln \tilde{X}_2 = \sum_j y_{2j} d \ln \tilde{X}_j$$

- Where recall $y_{ij} \equiv \frac{(\tau_{ij}/\bar{A}_i)^{1-\sigma} \tilde{X}_j}{\sum_{j'} (\tau_{ij'}/\bar{A}_i)^{1-\sigma} \tilde{X}_{j'}} = \frac{(w_i \tau_{ij}/\bar{A}_i)^{1-\sigma} \tilde{X}_j}{\sum_{j'} (w_i \tau_{ij'}/\bar{A}_i)^{1-\sigma} \tilde{X}_{j'}}$
- There are "ripple" effects of the improvement in access ($(1 - \sigma)\, y_{12} d \ln \tau_{12} > 0$)
  - The immediate effects work through the exports of location 1 (since this is about improving export costs of 1 to 2) as it can be seen from the first equation
  - But there are ripple effects as the exports of location 2 also adjust, as it can be seen from the second equation

Tracking Down the Ripple Effects

- Let us simplify things further and normalize $\tilde{X}_1 = 1 \implies d \ln \tilde{X}_1 = 0$

- Then we have

$$(\sigma - 1)\, d \ln W - y_{12} d \ln \tilde{X}_2 = (1 - \sigma)\, y_{12} d \ln \tau_{12}$$

and

$$(\sigma - 1)\, d \ln W = (y_{22} - 1)\, d \ln \tilde{X}_2$$

- The first equation tells that that an improvement in $\tau_{12} < 0$ tends to raise welfare but also to decline $\tilde{X}_2$ (as it makes 1 relatively richer)

- That is not the full story since welfare improvements, from the second equation, tend to lower $\tilde{X}_2$ further

Welfare Effects and the Role of the Network

- Combining the two equations we finally obtain

$$d \ln W = -\frac{(1 - y_{22})}{(1 - y_{22} + y_{12})} y_{12} d \ln \tau_{12} = - \underbrace{\frac{y_{21}}{y_{21} + y_{12}}}_{\text{network effects}} \underbrace{y_{12} d \ln \tau_{12}}_{\text{immediate impact}}$$

- This equation summarizes welfare effect of improvement in access of location 1 to location 2
  - Immediate impact larger if y_{12} is large (i.e if exports from 1 to 2 are large)
  - But for network effect y_{21} plays a key role (i.e eedback from 2 to 1)

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- Cool stuff!
  - This intuition will generalize (but math much harder) with many locations

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Total Differentiation of the Entire System

- Total differentiation of the entire system

$$\tilde{X}_i = W^{1-\sigma} \sum_j \left(\tau_{ij}/\bar{A}_i\right)^{1-\sigma} \tilde{X}_j \iff$$

$$\iff \ln \tilde{X}_i = (1-\sigma)\ln W + \ln \sum_j \left(\tau_{ij}/\bar{A}_i\right)^{1-\sigma} \tilde{X}_j$$

  - Now differentiate:

$$d\ln \tilde{X}_i - \sum_j \frac{(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j}{\sum_{j'}(\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}} d\ln \tilde{X}_j - (1-\sigma)\, d\ln W =$$

$$= (1-\sigma)\sum_j \frac{(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j}{\sum_{j'}(\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}} d\ln\left(\tau_{ij}/\bar{A}_i\right) \quad \text{for all } i$$

  - Use again $y_{ij} \equiv \frac{(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j}{\sum_{j'}(\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}} = \frac{(w_i\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j}{\sum_{j'}(w_i\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}}$, the ‘‘export share’’ of location j within the total exports of location i

Network of Locations: A Solvable System

- What do we learn from this expression? Consider

$$d \ln \tilde{X}_i - \sum_j y_{ij} d \ln \tilde{X}_j - (1-\sigma) \, d \ln W = (1-\sigma) \sum_j y_{ij} d \ln \left(\tau_{ij} / \bar{A}_i \right) \quad \text{for all } i$$

- Notice we have N equations and we normalize one variable (eg. $\tilde{X}_1 = 1 \implies d \ln \tilde{X}_1 = 0$) so we can solve for all $d \ln \tilde{X}_i$ for $i = 2, ..., N$ and $d \ln W$
  - Because of the linearity we can avoid using the aggregate labor clearing constraint
  - Basically the results are independent of the population level in the case $\alpha + \beta = 0$

- We will discuss now

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  - Because of the linearity we can avoid using the aggregate labor clearing constraint
  - Basically the results are independent of the population level in the case $\alpha + \beta = 0$
- We will discuss now
  - a. Intuition of how the network effects work in a spatial model
  - b. How to solve this system with linear algebra
  - c. How does this relate back to the case with two locations

Network Effects in Spatial Models

- What do we learn from this expression? Consider

$$d \ln \tilde{X}_i - \sum_j y_{ij} d \ln \tilde{X}_j + (\sigma - 1)\, d \ln W = (1-\sigma) \sum_j y_{ij} d \ln \left(\tau_{ij} / \bar{A}_i \right) \quad \text{for all } i$$

- Let us interpret the network effects of trade

$$\begin{aligned}
(\sigma - 1)\, d \ln W &= (1-\sigma) \sum_j y_{1j} d \ln \left(\tau_{1j} / \bar{A}_1 \right) + \sum_j y_{1j} d \ln \tilde{X}_j \\
d \ln \tilde{X}_2 &= (1-\sigma) \sum_j y_{2j} d \ln \left(\tau_{2j} / \bar{A}_2 \right) + \sum_j y_{2j} d \ln \tilde{X}_j - (\sigma - 1)\, d \ln W \\
&\cdots \\
d \ln \tilde{X}_N &= (1-\sigma) \sum_j y_{Nj} d \ln \left(\tau_{Nj} / \bar{A}_N \right) + \sum_j y_{Nj} d \ln \tilde{X}_j - (\sigma - 1)\, d \ln W
\end{aligned}$$

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Network Effects in Spatial Models

- What do we learn from this expression? Consider

$$d \ln \tilde{X}_i - \sum_j y_{ij} d \ln \tilde{X}_j + (\sigma - 1) d \ln W = (1 - \sigma) \sum_j y_{ij} d \ln \left(\tau_{ij} / \bar{A}_i \right) \quad \text{for all } i$$

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$$d \ln \tilde{X}_2 = (1 - \sigma) \sum_j y_{2j} d \ln \left(\tau_{2j} / \bar{A}_2 \right) + \sum_j y_{2j} d \ln \tilde{X}_j - (\sigma - 1) d \ln W$$

$$\cdots$$

$$d \ln \tilde{X}_N = (1 - \sigma) \sum_j y_{Nj} d \ln \left(\tau_{Nj} / \bar{A}_N \right) + \sum_j y_{Nj} d \ln \tilde{X}_j - (\sigma - 1) d \ln W$$

  - Start from a guess of $d \ln \tilde{X}_j$, $j = 2, ..., N$. Keep updating RHS with every iteration

Network Effects in Spatial Models

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$$d \ln \tilde{X}_i - \sum_j y_{ij} d \ln \tilde{X}_j + (\sigma - 1)\, d \ln W = (1-\sigma) \sum_j y_{ij} d \ln \left(\tau_{ij}/\bar{A}_i\right) \quad \text{for all } i$$

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$$(\sigma - 1)\, d \ln W = (1-\sigma) \sum_j y_{1j} d \ln \left(\tau_{1j}/\bar{A}_1\right) + \sum_j y_{1j} d \ln \tilde{X}_j$$

$$d \ln \tilde{X}_2 = (1-\sigma) \sum_j y_{2j} d \ln \left(\tau_{2j}/\bar{A}_2\right) + \sum_j y_{2j} d \ln \tilde{X}_j - (\sigma - 1)\, d \ln W$$

$$\ldots$$

$$d \ln \tilde{X}_N = (1-\sigma) \sum_j y_{Nj} d \ln \left(\tau_{Nj}/\bar{A}_N\right) + \sum_j y_{Nj} d \ln \tilde{X}_j - (\sigma - 1)\, d \ln W$$

  - Start from a guess of $d \ln \tilde{X}_j$, $j = 2, ..., N$. Keep updating RHS with every iteration
  - This sequence will converge to the true solution $d \ln W$, $d \ln \tilde{X}_j$

Solving the System with Linear Algebra

- Consider

$$(\sigma - 1)\, d \ln W + d \ln \tilde{X}_i - \sum_j y_{ij} d \ln \tilde{X}_j = (1 - \sigma) \sum_j y_{ij} d \ln \left(\tau_{ij} / \bar{A}_i\right) \quad \text{for all } i$$

- You can write this system in matrix form

$$\underbrace{\begin{bmatrix} \sigma - 1 & -y_{12} & \cdots & -y_{1N} \\ \sigma - 1 & 1 - y_{22} & \cdots & -y_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma - 1 & -y_{N2} & \cdots & 1 - y_{NN} \end{bmatrix}}_{\equiv M} \begin{bmatrix} d \ln W \\ d \ln \tilde{X}_2 \\ \vdots \\ d \ln \tilde{X}_N \end{bmatrix} = \begin{bmatrix} (1 - \sigma) \sum_j y_{1j} d \ln \frac{\tau_{1j}}{\bar{A}_1} \\ (1 - \sigma) \sum_j y_{2j} d \ln \frac{\tau_{2j}}{\bar{A}_2} \\ \vdots \\ (1 - \sigma) \sum_j y_{Nj} d \ln \frac{\tau_{Nj}}{\bar{A}_N} \end{bmatrix}$$

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  - How can you solve this system? Invert the M matrix!

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Solving the System with Linear Algebra

- Consider

$$(\sigma - 1)\, d\ln W + d\ln \tilde{X}_i - \sum_j y_{ij} d\ln \tilde{X}_j = (1-\sigma)\sum_j y_{ij} d\ln\left(\tau_{ij}/\bar{A}_i\right) \quad \text{for all } i$$

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  - How can you solve this system? Invert the M matrix!

$$\begin{bmatrix} d\ln W \\ \vdots \\ d\ln \tilde{X}_N \end{bmatrix} = M^{-1} \begin{bmatrix} (1-\sigma)\sum_j y_{1j} d\ln\left(\tau_{1j}/\bar{A}_1\right) \\ \vdots \\ (1-\sigma)\sum_j y_{Nj} d\ln\left(\tau_{Nj}/\bar{A}_N\right) \end{bmatrix}$$

Solving the System with Linear Algebra

- Consider

$$(\sigma - 1)\, d \ln W + d \ln \tilde{X}_i - \sum_j y_{ij} d \ln \tilde{X}_j = (1 - \sigma) \sum_j y_{ij} d \ln \left(\tau_{ij} / \bar{A}_i \right) \quad \text{for all } i$$

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- How can you solve this system? Invert the M matrix!

$$\begin{bmatrix} d\ln W \\ \vdots \\ d\ln \tilde{X}_N \end{bmatrix} = M^{-1} \begin{bmatrix} (1-\sigma)\sum_j y_{1j} d\ln\left(\tau_{1j}/\bar{A}_1\right) \\ \vdots \\ (1-\sigma)\sum_j y_{Nj} d\ln\left(\tau_{Nj}/\bar{A}_N\right) \end{bmatrix}$$

  - You can prove this matrix is always invertible (there is always a solution)
  - Matrix inverses can be written as infinite sums using Neumman series expansions
    - Using Neumman series expansions $(I-M)^{-1} = I + M + M^2 + \ldots$
    - Provides natural interpretation: M^n is the trade effects traced down through the network of locations (see Allen, Arkolakis, Takahashi '20)

Interpetation: Welfare Effect of Each ‘‘Shock’’

- Recall that

$$\begin{bmatrix} d\ln W \\ \vdots \\ d\ln \tilde{X}_N \end{bmatrix} = M^{-1} \begin{bmatrix} (1-\sigma)\sum_j y_{1j} d\ln(\tau_{1j}) \\ \vdots \\ (1-\sigma)\sum_j y_{Nj} d\ln(\tau_{Nj}) \end{bmatrix}$$

- Thus, with one 'shock' at a time (e.g. τ_{ij})

$$d\ln W = M_{1i}^{-1}(1-\sigma)\, y_{ij} d\ln(\tau_{ij})$$

- where M_{1i}^{-1} is the ith element of the 1st row of the inverse matrix

Back to the case of two locations

- Go back to our two locations. Log-linearize with respect to both τ and $\bar{A}$

$$(\sigma - 1)\, d \ln W = (1 - \sigma) \sum_{j=1,2} y_{1j} d \ln \left(\tau_{1j}/\bar{A}_1\right) + y_{12} d \ln \tilde{X}_2$$

$$(\sigma - 1)\, d \ln W = (1 - \sigma) \sum_{j=1,2} y_{2j} d \ln \left(\tau_{2j}/\bar{A}_2\right) - y_{21} d \ln \tilde{X}_2$$

- Equating the two and collecting terms

$$d \ln \tilde{X}_2 = (1 - \sigma) \left[\sum_{j=1,2} y_{2j} d \ln \left(\tau_{2j}/\bar{A}_2\right) - \sum_{j=1,2} y_{1j} d \ln \left(\tau_{1j}/\bar{A}_1\right) \right] / \left(y_{12} + y_{21}\right)$$

Substituting $d \ln \tilde{X}_2$ back to the first equation, we have

$$d \ln W = - \left[y_{21} \sum_{j=1,2} y_{1j} d \ln \left(\tau_{1j}/\bar{A}_1\right) + y_{12} \sum_{j=1,2} y_{2j} d \ln \left(\tau_{2j}/\bar{A}_2\right) \right] / \left(y_{12} + y_{21}\right)$$

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Back to the case of two locations: Recap

- Let us admire these two beautiful expressions

$$d\ln \tilde{X}_2 = (\sigma - 1)\left[\sum_{j=1,2} \frac{y_{1j}}{y_{12}+y_{21}} d\ln\left(\tau_{1j}/\bar{A}_1\right) - \sum_{j=1,2} \frac{y_{2j}}{y_{12}+y_{21}} d\ln\left(\tau_{2j}/\bar{A}_2\right)\right]$$

$$d\ln W = -\left[\sum_{j=1,2} \frac{y_{21}y_{1j}}{y_{12}+y_{21}} d\ln\left(\tau_{1j}/\bar{A}_1\right) + \sum_{j=1,2} \frac{y_{12}y_{2j}}{y_{12}+y_{21}} d\ln\left(\tau_{2j}/\bar{A}_2\right)\right]$$

  - Changes in adjusted market potential determined by relative exposure to shocks
  - Welfare increases for any improvement of spatial links or productivities!
  - Network effects discussed above formally related to inverse of M

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Back to the case of two locations: Recap

- Let us admire these two beautiful expressions

$$d \ln \tilde{X}_2 = (\sigma - 1) \left[\sum_{j=1,2} \frac{y_{1j}}{y_{12} + y_{21}} d \ln \left(\tau_{1j} / \bar{A}_1 \right) - \sum_{j=1,2} \frac{y_{2j}}{y_{12} + y_{21}} d \ln \left(\tau_{2j} / \bar{A}_2 \right) \right]$$

$$d \ln W = - \left[\sum_{j=1,2} \frac{y_{21} y_{1j}}{y_{12} + y_{21}} d \ln \left(\tau_{1j} / \bar{A}_1 \right) + \sum_{j=1,2} \frac{y_{12} y_{2j}}{y_{12} + y_{21}} d \ln \left(\tau_{2j} / \bar{A}_2 \right) \right]$$

  - Changes in adjusted market potential determined by relative exposure to shocks
  - Welfare increases for any improvement of spatial links or productivities!
  - Network effects discussed above formally related to inverse of M
    - $M = \begin{bmatrix} \sigma - 1 & -y_{12} \\ \sigma - 1 & 1 - y_{22} \end{bmatrix} = \begin{bmatrix} \sigma - 1 & -y_{12} \\ \sigma - 1 & y_{21} \end{bmatrix} \Longrightarrow$

$$M^{-1} = \frac{(\sigma - 1)^{-1}}{y_{21} + y_{12}} \begin{bmatrix} y_{21} & y_{12} \\ 1 - \sigma & \sigma - 1 \end{bmatrix}$$

Roadmap

- Solving for the Spatial Equilibrium Without Network Effects
- Improving Access and Geography: The Case of two Locations
- Effects of Improving Access on the Network of Locations
- Network Effects Analyzed
- **Iterative Solution of the Equilibrium**

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Iterative Solution of the Equilibrium

- Problem: How do we solve for the allocation of labor?

Iterative Solution of the Equilibrium

- Problem: How do we solve for the allocation of labor?
  - Take advantage of basic linear algebra results (Allen Arkolakis '14)
  - Spectacularly so exploit network to iterate from guess to arrive at true solution

- To make this clear rewrite the equilibrium equation,

$$L_i^{\frac{\sigma-1}{2\sigma-1}} = \bar{W}^{1-\sigma} \sum_j \tau_{ij}^{1-\sigma} L_j^{\frac{\sigma-1}{2\sigma-1}} \iff l_i = \lambda \sum_j K_{ij} l_j$$

where we have that $\lambda \equiv W^{\bar{1}-\sigma},\ K_{ij} \equiv \tau_{ij}^{1-\sigma},\ l(i) \equiv L_i^{\frac{\sigma-1}{2\sigma-1}}$

- Define $P_{ij} \equiv \frac{\sum_j K_{ij}}{\sum_i \sum_j K_{ij}}$ by normalizing K_{ij} matrix. **P** is a "transition" matrix
  - Intuition: P_{ij} is "probability" that location j trades with location i

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Iterative Solution to Equilibrium in Network Model

- In particular, we can simply reapply the transition matrix over and over again i.e. $\mathbf{P}^{(2)} = \mathbf{PP}$, $\mathbf{P}^{(3)} = \mathbf{PP}^{(2)}$... etc.
  - In the limit
    $$\mathbf{l}^* = \lim_{T \to \infty} P^{(T)} \mathbf{l}^{(0)}$$
    where $\mathbf{l}^{(0)}$ is some guess for the distribution of l_i's, say uniform, and $\mathbf{l}^*$ is the equilibrium distribution.

- Each application of the transition matrix represents a step, T, in the system.
  - It can be proven that in limit, $T \to \infty$, iteration convergences to the **true** solution
  - Truly amazing result: think of thousands of locations. A random guess will suffice

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- Let's examine this convergence!

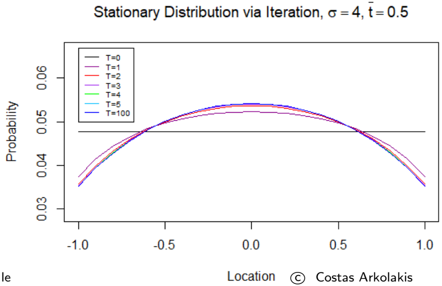
Iterative Solution to Equilibrium in Network Model

- We apply this to the model from Lectures 7-8. Again written as:

$$L_i^{\frac{\sigma-1}{2\sigma-1}} = \bar{W}^{1-\sigma} \sum_j \tau_{ij}^{1-\sigma} L_j^{\frac{\sigma-1}{2\sigma-1}}$$

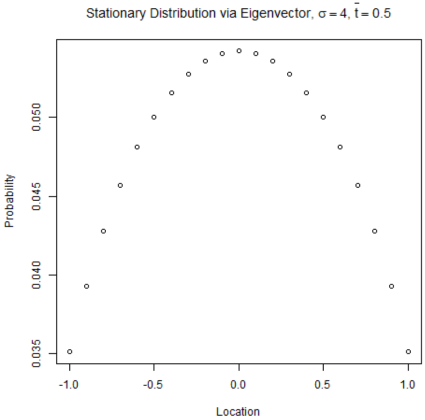
  - (No spillovers: $\alpha = \beta = 0$ and $\bar{A}_i, \bar{u}_i = 1$)
  - Trade costs given by $\tau_{ij} = e^{\bar{t}|i-j|}$ with and i, j on the line from $[-1, 1]$

- Notice how fast the convergence happens!



Solution to Equilibrium in Network Model

- This iteration gives the same solution as our analytical one (cosine)



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