# The Economics of Space 433: Section 1

## An Introduction to Producer Problem and Dixit-Stiglitz Preferences

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# Logistics

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- TF session: Friday 9:30 am, 10:30 am
- Office Hour: Friday 11:30 am or by appointments

# Consumer and Producer Problem: Roadmap

- Two major economic agents: **consumers** and **producers** (firms)
- We usually first solve for each agent's optimization problem
  - Consumer's utility maximization $\rightarrow$ demand
  - Producer's profit maximization $\rightarrow$ supply
  - Then, find an equilibrium by finding prices that clear the markets
- Today:
  - Review producer's problem
  - Analyze the consumer problem: Cobb-Douglas, Dixit-Stigliz

# Producer Problem

- Producer maximizes the **profit** $\pi$ subject to **production technology** $Y$ by choosing optimal **inputs** given **prices (demand)**

- Firm's profit = revenues - production costs
- Firm's output $Y$ is determined by inputs
  - E.g. Linear technology using labor $L$: $Y = AL$
  - E.g. Cobb-Douglas production function with capital $K$ and labor $L$: $Y = AK^{\alpha}L^{1-\alpha}$

- Assume there are two inputs, capital $K$ and labor $L$

- Also assume that firms take goods price $p$ and input prices $(w, r)$ as given (competitive market)

# Producer Problem: FOC

- Firm maximizes the profit by choosing optimal level of inputs $(K, L)$

$$\max_{K,L} \quad \underbrace{pY(K,L)}_{\text{revenue}} - \underbrace{(rK + wL)}_{\text{production costs}}$$

- Take the FOC:

$$p\frac{\partial Y(K,L)}{\partial K} = r$$
$$p\frac{\partial Y(K,L)}{\partial L} = w$$

# Producer Problem: FOC

- To maximize profit, firm equalizes the marginal revenue to marginal cost of each input
- Marginal revenue (MR) is an increase of revenue w.r.t. an increase of an input
  - Hiring one more worker $\rightarrow$ increase output, and thus revenue
  - Firms take good prices as given $\rightarrow$ MR is the price $p$ multiplied by marginal product (MP) $\frac{\partial Y}{\partial K}$ ($\frac{\partial Y}{\partial L}$)
- Marginal cost (MC) is an increase of production cost w.r.t. an increase of input
  - Hiring one more worker $\rightarrow$ need to pay more wages
  - Firms take input prices as given $\rightarrow$ MC is price of an input $r$ ($w$)

# Producer Problem: Example

- Now, suppose $Y(L) = AL$. Then, firm's problem becomes

$$\max_{L} \underbrace{p(AL)}_{\text{revenue}} - \underbrace{wL}_{\text{cost}}$$

- FOC:

$$pA = w$$

  - If $pA > w$, firms will produce infinite amount of output
  - If $pA < w$, firms will produce zero output
  - This usually cannot be an equilibrium since the price will adjust in the goods market

- cf) Rosen-Roback: the economy with agglomeration forces $A = \overline{A}L^{\alpha}$

# Consumer Problem

- Consumer maximizes **utility** subject to **the budget constraint** choosing optimal **consumption** given **prices**

- Commonly used utility function: Cobb-Douglas (C-D)
  - Consumer's utility function $U(c_1, c_2, ..., c_N) = \prod_{i=1}^{N} c_i$
  - $c_i$ = the quantities consumed of each good
  - Budget constraint: $p_1 c_1 + p_2 c_2 + .. + p_N c_N = w$

- In this class, we focus on Dixit-Stigliz (D-S) utility
  - Consumer's utility function
  $$U(c_1, c_2, ..., c_N) = \left( c_1^{\frac{\sigma-1}{\sigma}} + c_2^{\frac{\sigma-1}{\sigma}} + ... + c_N^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$
  - $\sigma$ captures how easily consumers can *substitute* between two goods
  - If $\sigma$ is large, consumers can easily substitute between $c_i$ and $c_j$
  - E.g. If $\sigma \to \infty$, then $U(c_1, c_2, ...c_N) = c_1 + c_2 + ... + c_N$

# C-D: the Set-up

- Consumer's problem:

$$\max_{c_1, c_2} c_1^{\alpha} c_2^{1-\alpha}$$

$$\text{subject to } p_1 c_1 + p_2 c_2 = w$$

- From the consumer's perspective, $(p_1, p_2, w, \alpha)$ are **exogenous** variables/parameters
  - They are just "given" to the consumer
- On the other hand, $c_1, c_2$ are **endogenous** variables
  - The consumer *chooses* their value in order to maximize utility
  - We denote the optimal choice as $(c_1^*, c_2^*)$ and the corresponding maximized level of utility as $U^*$

# C-D: Lagrangian and FOCs

- To solve the maximization problem, write the Lagrangean function

$$\mathcal{L} = c_1^{\alpha} c_2^{1-\alpha} + \mu \left( w - p_1 c_1 - p_2 c_2 \right)$$

where $\mu$ is the Lagrange multiplier

- Take the first-order condition (FOC)

$$\frac{\partial \mathcal{L}}{\partial \mathrm{c}_1} = \alpha c_1^{\alpha-1} c_2^{1-\alpha} - \mu p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mathrm{c}_2} = (1-\alpha) c_1^{\alpha} c_2^{-\alpha} - \mu p_2 = 0$$

$$\text{BC} : w - \left( p_1 c_1 + p_2 c_2 \right) = 0$$

# C-D: Algebra 1

- Rewrite first 2 equations

$$\alpha c_1^{\alpha-1} c_2^{1-\alpha} = \mu p_1$$
$$(1-\alpha) c_1^{\alpha} c_2^{-\alpha} = \mu p_2$$

- Divide the first by the second

$$\frac{\alpha c_1^{\alpha-1} c_2^{1-\alpha}}{(1-\alpha) c_1^{\alpha} c_2^{-\alpha}} = \frac{\mu p_1}{\mu p_2} \Longrightarrow$$

$$\frac{\alpha c_2}{(1-\alpha) c_1} = \frac{p_1}{p_2} \Longrightarrow$$

$$c_2 = \frac{(1-\alpha) p_1 c_1}{\alpha p_2} \tag{1}$$

# C-D: Algebra 2

- Now substitute $c_2$ in the budget constraint

$$p_1 c_1 + p_2 \frac{(1-\alpha) p_1 c_1}{\alpha p_2} = w \Longrightarrow$$

$$p_1 c_1 \left(1 + \frac{(1-\alpha)}{\alpha}\right) = w \Longrightarrow$$

$$c_1^* = \frac{\alpha w}{p_1} \tag{2}$$

- We have expressed $c_1$ as a function of exogenous parameters only! Thus, we've found the utility-maximizing choice, $c_1^*$
- To find $c_2^*$, substitute $c_1^*$ into (1),

$$c_2^* = \frac{(1-\alpha) p_1}{\alpha p_2} \frac{\alpha w}{p_1} = \frac{(1-\alpha) w}{p_2}$$

# C-D: Solution, Utility

- We have found the optimal consumption:

$$(c_1^*, c_2^*) = \left(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\right)$$

- To find the ''optimized'' utility level, plug $c_1^*, c_2^*$ into the utility function:

$$U^* = U(c_1^*, c_2^*) = (c_1^*)^\alpha (c_2^*)^{1-\alpha}$$

$$\Longrightarrow U^* = \left(\frac{\alpha w}{p_1}\right)^\alpha \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha} = \frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}} w$$

$$\Longrightarrow U^* = \frac{w}{\frac{p_1^\alpha p_2^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}}} \tag{3}$$

# C-D: Price Index

- Define a new variable $P = \frac{p_1^{\alpha} p_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$, the maximized utility $U^*$ is simply:

$$U^* = \frac{w}{\frac{p_1^{\alpha} p_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}} = \frac{w}{P}$$

- $P$ is known as the **price index**
  - $P$ is increasing in $p_1$ and $p_2$
  - $P$ resembles C-D utility function
    - Precise form of the price index is influenced by the underlying utility function
    - Different utility function would give the index

# C-D: Expenditure Shares

- Finally, let's compute the **expenditure shares** of each good
  - That is, the fraction of the consumer's income that is spent on each good
  - E.g. expenditure share of good 1 is: $\lambda_1 = \frac{\text{money spent on good 1}}{\text{total income}}$
- Total money spent on good $i$ is $p_i c_i^*$.
  So, expenditure share of good $i$ is $\lambda_i = \frac{p_i c_i^*}{w}$
- From $c* = \left(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\right)$, we get

$$\lambda_1 = \alpha$$
$$\lambda_2 = 1 - \alpha$$

- Note that expenditure share is independent of prices or income
  - E.g. if $p_1 \uparrow$, buys fewer units of $c_1$ but spends more money on each unit
  - These two effects cancel out so that expenditure share stays constant

# Moving on to D-S

- Having shown the facts and properties of consumer choice under C-D preferences, we now move on to **Dixit-Stiglitz** preferences

- We will use the exact same sequence of steps, presenting conditions for CD and DS side-by-side to make it even more explicit

# C-D vs. D-S: Set-Up

- Consumer's utlity-maximization problem:

$$\max_{c_1, c_2} U(c_1, c_2) \quad \text{subject to } p_1 c_1 + p_2 c_2 = w$$

  - The same consumer's optimization problem
  - Only difference is the functional form of the utility $U(c_1, c_2)$

**Cobb-Douglas case**

$$\max_{c_1, c_2} c_1^{\alpha} c_2^{1-\alpha}$$

subject to:

$$p_1 c_1 + p_2 c_2 = w$$

**Dixit-Stiglitz case**

$$\max_{c_1, c_2} \left( c_1^{\frac{\sigma-1}{\sigma}} + c_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

subject to:

$$p_1 c_1 + p_2 c_2 = w$$

# C-D vs. D-S: Lagrangean and FOCs

Write the Lagrangian and take first-order conditions:

**Cobb-Douglas case**

$$\mathcal{L} = c_1^{\alpha} c_2^{1-\alpha} + \mu(w - p_1 c_1 - p_2 c_2)$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = \alpha c_1^{\alpha-1} c_2^{1-\alpha} - \mu p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = (1-\alpha) c_1^{\alpha} c_2^{-\alpha} - \mu p_2 = 0$$

**Dixit-Stiglitz case**

$$\mathcal{L} = \left( c_1^{\frac{\sigma-1}{\sigma}} + c_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} + \mu(w - p_1 c_1 - p_2 c_2)$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{\sigma}{\sigma-1} \left[ c_1^{\frac{\sigma-1}{\sigma}} + c_2^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}-1} \frac{\sigma-1}{\sigma} c_1^{\frac{\sigma-1}{\sigma}-1} - \mu p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{\sigma}{\sigma-1} \left[ c_1^{\frac{\sigma-1}{\sigma}} + c_2^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}-1} \frac{\sigma-1}{\sigma} c_2^{\frac{\sigma-1}{\sigma}-1} - \mu p_2 = 0$$

# C-D vs. D-S: Algebra 1

Algebraically manipulating equations:

**Cobb-Douglas case**

$$\alpha c_1^{\alpha-1} c_2^{1-\alpha} = \mu p_1$$

$$(1-\alpha) c_1^{\alpha} c_2^{-\alpha} = \mu p_2$$

**Dixit-Stiglitz case**

$$[c_1^{\frac{\sigma-1}{\sigma}} + c_2^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}-1} c_1^{\frac{\sigma-1}{\sigma}-1} = \mu p_1$$

$$[c_1^{\frac{\sigma-1}{\sigma}} + c_2^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}-1} c_2^{\frac{\sigma-1}{\sigma}-1} = \mu p_2$$

Dividing the top equation by the bottom equation, we get:

$$\frac{\alpha c_2}{(1-\alpha) c_1} = \frac{p_1}{p_2} \Rightarrow$$

$$c_2 = \frac{(1-\alpha) p_1}{\alpha p_2} c_1$$

$$\frac{c_1^{\frac{\sigma-1}{\sigma}-1}}{c_2^{\frac{\sigma-1}{\sigma}-1}} = \frac{p_1}{p_2} \Rightarrow$$

$$c_2 = (\frac{p_2}{p_1})^{-\sigma} c_1$$

# C-D vs. D-S: Algebra 2

Substituting $c_2$ into budget constraints, we get $c_1^*$:

**Cobb-Douglas case**

$$p_1 c_1 + p_2 \left( \frac{(1-\alpha)p_1}{\alpha p_2} c_1 \right) = w \Rightarrow$$

$$c_1^* = \frac{\alpha w}{p_1}$$

**Dixit-Stiglitz case**

$$p_1 c_1 + p_2 \left( (\frac{p_2}{p_1})^{-\sigma} c_1 \right) = w \Rightarrow$$

$$c_1^* = \frac{p_1^{-\sigma} w}{p_1^{1-\sigma} + p_2^{1-\sigma}}$$

Now plug it back into $c_2$ equations:

$$c_2^* = \frac{(1-\alpha)w}{p_2}$$

$$c_2^* = \frac{p_2^{-\sigma} w}{p_1^{1-\sigma} + p_2^{1-\sigma}}$$

# C-D vs. D-S: Solution, Utility

Thus, the utility-maximizing choice bundle is:

**Cobb-Douglas case**

$$(c_1^*, c_2^*) = \left( \frac{\alpha w}{p_1}, \frac{(1-\alpha) w}{p_2} \right)$$

**Dixit-Stiglitz case**

$$(c_1^*, c_2^*) = \left( \frac{p_1^{-\sigma} w}{p_1^{1-\sigma} + p_2^{1-\sigma}}, \frac{p_2^{-\sigma} w}{p_1^{1-\sigma} + p_2^{1-\sigma}} \right)$$

We can then find the achieved level of utility $U^* = U(c_1^*, c_2^*)$:

$$\begin{aligned} U^* &= (c_1^*)^{\alpha} (c_2^*)^{1-\alpha} \\ &= \frac{w}{\frac{p_1^{\alpha} p_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}} \end{aligned}$$

$$\begin{aligned} U^* &= [(c_1^*)^{\frac{\sigma-1}{\sigma}} + (c_2^*)^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} \\ &= \left[ \left( \frac{p_1^{-\sigma} w}{p_1^{1-\sigma} + p_2^{1-\sigma}} \right)^{\frac{\sigma-1}{\sigma}} + \left( \frac{p_2^{-\sigma} w}{p_1^{1-\sigma} + p_2^{1-\sigma}} \right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \\ &= \frac{w}{\left( p_1^{1-\sigma} + p_2^{1-\sigma} \right)^{\frac{1}{1-\sigma}}} \end{aligned}$$

# C-D vs. D-S: Price Index

Let's define the **price index**, $P$, so that we can express utility as $U^* = \frac{w}{P}$:

**Cobb-Douglas case**

$$P \equiv \frac{p_1^{\alpha} p_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$$

**Dixit-Stiglitz case**

$$P \equiv \left(p_1^{1-\sigma} + p_2^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

Note that in both cases:

- $P$ increases in either $p_1$ or $p_2$
- $P$ bears some similarities to the utility function
  - Once more, the form of the index is influenced by the underlying utility function

# C-D vs. D-S: Expenditure Shares

Substituting the values of $c_i^*$:

**Cobb-Douglas case**

$$\lambda_1 = \alpha$$
$$\lambda_2 = 1 - \alpha$$

**Dixit-Stiglitz case**

$$\lambda_1 = \frac{p_1^{1-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}$$
$$\lambda_2 = \frac{p_2^{1-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}$$

- Expenditure shares are affected by prices in D-S case, but not in C-D case
- In D-S case, a higher $p_1$ decreases a spending share of good 1 ($\sigma > 1$)
  - Consumers can more easily substitute toward good 2 than C-D case
  - ''Fewer units'' effect dominates the ''higher price per unit'' effect
- For both cases, $\lambda_1 + \lambda_2 = 1$
  - Recall BC: $\sum_i \lambda_i = \sum_i \frac{p_i c_i}{w} = 1$

# D-S: Expenditure Shares

- Note the structure of expenditure shares:

$$\lambda_i = \frac{p_i^{1-\sigma}}{\sum_{i'} p_{i'}^{1-\sigma}}$$

  - It has the structure $\frac{K_{\text{"that good"}}}{K_{\text{"summed across all goods"}}}$
  - Moreover, the denominator is a function of the price index, i.e. $P^{1-\sigma}$

- It turns out that this structure generalizes to N goods: $\lambda_i = \frac{p_i^{1-\sigma}}{\sum_{s=1}^{N} p_s^{1-\sigma}}$

# Appendix: D-S and limit cases

- We already checked that the limit of D-S with $\sigma \to \infty$

$$\lim_{\sigma \to \infty} (\alpha c_1^{\frac{\sigma-1}{\sigma}} + (1-\alpha) c_2^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} = \alpha c_1 + (1-\alpha) c_2$$

- In fact, C-D is the limit of D-S with $\sigma \to 1$

$$\lim_{\sigma \to 1} (\alpha c_1^{\frac{\sigma-1}{\sigma}} + (1-\alpha) c_2^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} = c_1^{\alpha} c_2^{1-\alpha}$$