The Economics of Space 433: Section 3

Dixit-Stiglitz utility and hotelling model

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General Dixit-Stiglitz utility

- Consider a general Dixit-Stiglitz utility:

$$U(\{c_i\}_{i\in S}) = \left(\sum_{i\in S} \alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

  where $\sigma > 0, \alpha_i > 0, \forall i$.

  1. Multiple goods $S = \{1, 2, ..., I\}$
  2. Preferences across goods $\alpha_i^{\frac{1}{\sigma}}$

- If $S = \{1, 2\}, \alpha_i = 1$, we go back to TF session 1 example
- If $\alpha_i = 1$, we go back to utility function in lecture slides

Consumer Optimization

- Consumers maximize utility $U(\{c_i\}_{i\in S})$ s.t. $\sum_{i\in S} c_i p_i \leq w$
- Setup the Lagrangian function

$$\mathcal{L} = \left(\sum_{i\in S} \alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} + \mu\left(w - \sum_{i\in S} p_i c_i\right).$$

- FOC: differentiating it with respect to c_i,

$$\left(\frac{\sigma}{\sigma-1}\right)\left(\sum_{i\in S} \alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} \alpha_i^{\frac{1}{\sigma}}\left(\frac{\sigma-1}{\sigma}\right)(c_i)^{\frac{\sigma-1}{\sigma}-1} = \mu p_i$$

$$\Rightarrow \alpha_i^{\frac{1}{\sigma}}\left(\sum_{i\in S} \alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} c_i^{-\frac{1}{\sigma}} = \mu p_i$$

Consumer Optimization

- Observe that we have two common terms that are independent of i

$$\alpha_i^{\frac{1}{\sigma}} \left(\sum_{i \in S} \alpha_i^{\frac{1}{\sigma}} c_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} c_i^{-\frac{1}{\sigma}} = \mu p_i$$

- Take the ratio of two goods i and k:

$$\left(\frac{\alpha_i}{\alpha_k} \right)^{\frac{1}{\sigma}} \left(\frac{c_i}{c_k} \right)^{-\frac{1}{\sigma}} = \frac{p_i}{p_k}$$

$$\Rightarrow c_i = \frac{\alpha_i}{\alpha_k} \left(\frac{p_i}{p_k} \right)^{-\sigma} c_k$$

- Plug this into BC:

$$w = \sum_{i \in S} c_i p_i = \sum_{i \in S} \frac{\alpha_i}{\alpha_k} \left(\frac{p_i}{p_k} \right)^{-\sigma} c_k p_i = \frac{p_k^{\sigma}}{\alpha_k} c_k \sum_{i \in S} \alpha_i p_i^{1-\sigma}$$

Consumer Optimization

- ... which gives

$$w = \frac{p_k^{\sigma}}{\alpha_k} c_k \sum_{i \in S} \alpha_i p_i^{1-\sigma}$$

$$\Rightarrow c_k = \frac{\alpha_k p_k^{-\sigma}}{\sum_{i \in S} \alpha_i p_i^{1-\sigma}} w$$

- Define **Price index**

$$P = \left(\sum_{i \in S} \alpha_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

Q) How does it depend on corr(α_i, p_i)?

Consumer Optimization

- Combine c_k and P:

$$c_k = \frac{\alpha_k p_k^{-\sigma}}{\sum_{i\in S} \alpha_i p_i^{1-\sigma}} w$$

$$P = \left(\sum_{i\in S} \alpha_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

- ... to get **Spending share** λ_i:

$$\lambda_k = \frac{c_k p_k}{w} = \alpha_k \frac{p_k^{1-\sigma}}{P^{1-\sigma}}$$

Q) How does it depend on α_k, p_k, w, P?
Q) What if $\sigma = 1$?

Consumer Optimization

- Consumer **value** becomes C:

$$
\begin{aligned}
C = U(\{c_i^*\}_{i \in S}) &= \left(\sum_{i \in S} \alpha_i^{\frac{1}{\sigma}} \left(\frac{\alpha_i p_i^{-\sigma} w}{P^{1-\sigma}} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \\
&= \left(\sum_{i \in S} \alpha_i \frac{p_i^{1-\sigma} w^{\frac{\sigma-1}{\sigma}}}{P^{(1-\sigma)\frac{\sigma-1}{\sigma}}} \right)^{\frac{\sigma}{\sigma-1}} \\
&= \left(\sum_{i \in S} \alpha_i p_i^{1-\sigma} \right)^{\frac{\sigma}{\sigma-1}} \frac{w}{P^{1-\sigma}} \\
&= \frac{w}{P}
\end{aligned}
$$

Q) How does it depend on w, P and individual p_i, α_i?

Hotelling Model (PS1 Q4)

- Uniformly distributed consumers on [0,1] have a unitary demand
- Linear transportation cost t
- Two firms selling a homogeneous good for free
- Social planner minimizes the total transportation cost of the system with respect to the location of the firms

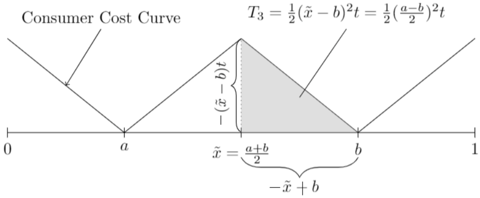
- Let's first compute the total transportation cost given two locations (a, b)

Hotelling Model (PS1 Q4)

- A consumer who is at $\tilde{x} = \frac{a+b}{2}$ is indifferent between two firms
  - Consumers at $x < \frac{a+b}{2}$ choose a firm at a and others choose a firm at b

- The total cost can be calculated as the area of triangles in the Figure below

$$\frac{1}{2}a^2 t + \frac{1}{2}(\frac{a-b}{2})^2 t + \frac{1}{2}(\frac{a-b}{2})^2 t + \frac{1}{2}(1-b)^2 t$$



Hotelling Model (PS1 Q4)

- SP problem:

$$\min_{a,b} \frac{1}{2}a^2 t + \frac{1}{2}(\frac{a-b}{2})^2 t + \frac{1}{2}(\frac{a-b}{2})^2 t + \frac{1}{2}(1-b)^2 t$$

- FOC:

$$at + \frac{a-b}{2}t = 0$$

$$-\frac{a-b}{2}t - (1-b)t = 0$$

- First condition gives $b = 3a$. Plugging this result into the second condition, we have $(a - 1 + 3a)t = 0$, which gives $a = \frac{1}{4}$ and $b = \frac{3}{4}$.