The Economics of Space 433: Section 3

Aggregation of the Spatial Equilibrium

Costas Arkolakis and Eduardo Fraga[1]

[1]Yale University

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General Equilibrium

- Our goal in these slides is to derive the spatial model's two General Equilibrium equations:
  - These are the equations that determine equilibrium wages and the equilibrium spatial allocation of labor.
- We start from the equations that represent agents' optimal choices:
  - Consumer Optimization
  - Firm's Profit Maximization
- Then feed these agent-optimization equations into the two equations that "close the model":
  - Feasibility/market clearing
  - Welfare equalization
- Finally, algebraically manipulate ensuing expressions to obtain the two GE equations!

Consumer Optimization

- Consumer in location j maximizes utility subject to budget constraint:
  - Takes prices $\{p_{ij}\}_i$ and wage w_j as given.
- Let's define the share of location i in location j's spending as:

$$\lambda_{ij} \equiv \frac{p_{ij} c_{ij}}{\sum_k p_{kj} c_{kj}} = \frac{p_{ij} c_{ij}}{w_j}$$

- After maximizing consumer's utility, one can show that spending shares satistfy the following equation:

$$\lambda_{ij} = \frac{p_{ij}^{1-\sigma}}{P_j^{1-\sigma}} \tag{1}$$

  where $P_j^{1-\sigma} = \sum_k p_{kj}^{1-\sigma}$.
- One can also show that real consumption can be written in this simple way:

$$C_j = \frac{w_j}{P_j} \tag{2}$$

Firm Optimization

- Firm's productive technology in location i is given by:

$$y_i = A_i L_i \tag{3}$$

  - Firm must also pay transportation cost: to sell one unit of the good to location j, firm must ship τ_{ij} units.

- Thus, firm's marginal cost of delivering an extra unit of the good to location j is given by: $\frac{\tau_{ij} w_i}{A_i}$.

- We have perfect competition, so in equilibrium the price charged to the consumer must equal the marginal cost:

$$p_{ij} = \underbrace{\frac{w_i}{A_i}}_{\text{production cost}} \times \underbrace{\tau_{ij}}_{\text{shipping cost}} \tag{4}$$

Consumer + Firm

- Substituting the firm's pricing equation (4) into the consumer's spending-share equation (1), we obtain:

$$\lambda_{ij} = \frac{p_{ij}^{1-\sigma}}{P_j^{1-\sigma}} = \frac{p_{ij}^{1-\sigma}}{\sum_k p_{kj}^{1-\sigma}} = \frac{\left(\tau_{ij} w_i / A_i\right)^{1-\sigma}}{\sum_k \left(\tau_{kj} w_k / A_k\right)^{1-\sigma}} \tag{5}$$

- ... or, more compactly:

$$\lambda_{ij} = \left(\frac{\tau_{ij} w_i}{A_i P_j}\right)^{1-\sigma} \tag{6}$$

with $P_j^{1-\sigma} = \sum_k \left(\tau_{kj} w_k / A_k\right)^{1-\sigma}$.

Consumer + Firm (cont.)

- Given these spending shares, we can also write the value (in dollars) of total aggregate exports from i to j as follows:

$$X_{ij} = \lambda_{ij} Y_j \tag{7}$$

where Y_j is total aggregate income of citizens of location j.

- The only kind of income received by citizens of location j are wages, thus $Y_j = w_j L_j$.

Closing the Model

- To "close" the model, we impose three conditions.
- First is "feasibility" or "market clearing":
  - The income received by location i must equal its worldwide sales:

$$\underbrace{Y_i}_{\text{income of } i} = w_i L_i = \underbrace{\sum_j X_{ij}}_{\text{sales to all destinations from } i} = \sum_j \lambda_{ij} Y_j = \sum_j \lambda_{ij} w_j L_j \tag{8}$$

- Second is welfare equalization:
  - Given free mobility, welfare must be the same for all locations:

$$\bar{W} = W_i, \ \forall i \tag{9}$$

- Third is that sum of population across locations must equal (exogenously given) total population:

$$\bar{L} = \sum_i L_i \tag{10}$$

Deriving GE1

- Let's now derive the first General Equilibrium equation (GE1).
- Using the feasibility constraint (8) and substituting the solution for λ_{ij} from equation (6), we get:

$$w_i L_i = \sum_j \lambda_{ij} w_j L_j \iff w_i L_i = \sum_j \left(\frac{\tau_{ij} w_i}{P_j A_i}\right)^{1-\sigma} w_j L_j$$

$$\Rightarrow w_i^{\sigma} L_i = \sum_j \tau_{ij}^{1-\sigma} P_j^{\sigma-1} A_i^{\sigma-1} w_j L_j$$

Deriving GE1 (cont.)

- We want to get rid of the price index P_j. How can we do it?
  - Remember: local welfare in j can be written as $W_j = C_j u_j$.
  - By equation (2), we have $C_j = \frac{w_j}{P_j}$. Thus:

$$W_j = \frac{w_j}{P_j} u_j \Rightarrow P_j = \frac{w_j}{W_j} u_j \tag{11}$$

- Substituting for P_j in the feasibility constraint, we then get:

$$w_i^\sigma L_i = \sum_j W_j^{1-\sigma} \tau_{ij}^{1-\sigma} w_j^{\sigma-1} u_j^{\sigma-1} A_i^{\sigma-1} w_j L_j$$

$$\Rightarrow w_i^\sigma L_i = \sum_j W_j^{1-\sigma} \tau_{ij}^{1-\sigma} u_j^{\sigma-1} A_i^{\sigma-1} w_j^\sigma L_j \tag{12}$$

Deriving GE1 (cont.)

- Now let us do the following:
  - Use welfare equalization to substitute $\bar{W}$ for W_j, ...
  - ... substitute $A_i = \bar{A}_i L_i^{\alpha}$ and $u_j = \bar{u}_j L_j^{-\beta}$, ...
  - ... and do some boring algebra.
- Equation (12) then becomes:

$$w_i^{\sigma} L_i^{1-\alpha(\sigma-1)} = \bar{W}^{1-\sigma} \sum_j \left(\bar{A}_i\right)^{(\sigma-1)} \left(\bar{u}_j\right)^{\sigma-1} \left(\tau_{ij}\right)^{1-\sigma} w_j^{\sigma} L_j^{1-\beta(\sigma-1)} \tag{13}$$

- This is the first GE equation (GE1)!

Deriving GE2

- Deriving the second general equilibrium equation (GE2) is more straightforward.
- As we saw above, the price index in j can be written as:

$$P_j^{1-\sigma} = \sum_k (\tau_{kj} w_k / A_k)^{1-\sigma}$$

- Playing around with subscripts (which are arbitrary to begin with), we can rewrite:

$$P_i^{1-\sigma} = \sum_j (\tau_{ji} w_j / A_j)^{1-\sigma}$$

- From equation (11), we have $P_i = \frac{w_i}{W_i} u_i$.
  - Substituting for P_i in the price index equation, we get:

$$w_i^{1-\sigma} = W_i^{1-\sigma} u_i^{\sigma-1} \sum_j (\tau_{ji} w_j / A_j)^{1-\sigma} \tag{14}$$

Deriving GE2 (cont.)

- Substituting welfare equalization ($W_j = \bar{W}$) and $u_i = \bar{u}_i L_i^{-\beta}$ into equation (14) and manipulating a bit, we get:

$$L_i^{\beta(\sigma-1)} = w_i^{\sigma-1} \bar{u}_i^{\sigma-1} \bar{W}^{1-\sigma} \sum_j (\tau_{ji} w_j / A_j)^{1-\sigma} \Rightarrow$$

$$L_i = (w_i \bar{u}_i)^{\frac{1}{\beta}} \left(\sum_j (\tau_{ji} w_j / A_j)^{1-\sigma} \right)^{\frac{-1}{\beta(1-\sigma)}} \bar{W}^{-\frac{1}{\beta}} \tag{15}$$

- This is the second GE equation (GE2)!

The GE System

- Thus, we have derived the two GE equations, namely (13) and (15).
- Here are they:

$$w_i^{\sigma} L_i^{1-\alpha(\sigma-1)} = \bar{W}^{1-\sigma} \sum_j \left(\bar{A}_i\right)^{(\sigma-1)} \left(\bar{u}_j\right)^{\sigma-1} \left(\tau_{ij}\right)^{1-\sigma} w_j^{\sigma} L_j^{1-\beta(\sigma-1)}$$

$$L_i = \left(w_i \bar{u}_i\right)^{\frac{1}{\beta}} \left(\sum_j \left(\tau_{ji} w_j / A_j\right)^{1-\sigma} \right)^{\frac{-1}{\beta(1-\sigma)}} \bar{W}^{-\frac{1}{\beta}}$$

- It's a (non-linear) system of $2N$ unknowns (N wages and N populations) on $2N$ equations.
  - There's also one price normalization.

A Special Case

- The GE system of equations formed by equations (13) and (15) isn't generally easy to solve.
- But consider the subcase with no spillovers: $\alpha = \beta = 0$.
- Equation GE1 then simplifies to:

$$w_i^{\sigma} L_i = \bar{W}^{1-\sigma} \sum_j \underbrace{\left(\bar{A}_i\right)^{(\sigma-1)} \left(\bar{u}_j\right)^{\sigma-1} \left(\tau_{ij}\right)^{1-\sigma}}_{\text{constants}} w_j^{\sigma} L_j$$

A Special Case (cont.)

- And equation GE2 simplifies to:

$$1 = w_i^{\sigma-1}\bar{u}_i^{\sigma-1}\bar{W}^{1-\sigma}\sum_j (\tau_{ji}w_j/A_j)^{1-\sigma} \Rightarrow \left(\frac{w_i\bar{u}_i}{\bar{W}}\right)^{1-\sigma} = \sum_j (\tau_{ji}w_j/A_j)^{1-\sigma}$$

  - which by equation (11) implies: $P_i^{1-\sigma} = \sum_j (\tau_{ji}w_j/A_j)^{1-\sigma}$, which is simply our old price index expression.
  - Thus, equation GE2 doesn't give us any extra information.

- The new equation GE1 constitutes a **linear** system of equations (on variable $w_i^\sigma L_i$).
  - Standard results in linear algebra imply that it has an unique solution!