The Economics of Space 433: Section 5

Normalization and Log-Linearization

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Goals

- Our goal in these slides is twofold:
  - First, we will discuss why our workhorse spatial model features one price normalization
  - Second, we will work on the algebra of log-linearization.
- Price normalization:
  - Equilibrium system has one less equation than it looks (courtesy of Walras's Law)
  - We then impose one price normalization to "close" the system.
- Log-linearization:
  - Express shocks to the equilibrium system and the response of endogenous variables...
  - ... but in a compact way.

Roadmap

- **Normalization**

- Log-Linearization

The Equilibrium System

- Consider our workhorse spatial model with N locations
- Given exogenous variables $\{\tau_{ij}, \bar{A}_i, \bar{u}_i\}$ and parameters (α, β, σ), the equilibrium endogenous variables $\{L_i, w_i, P_i, \bar{W}\}$ are found by solving the following system of equations:

$$w_i L_i = \sum_j \left(\frac{\tau_{ij} w_i}{\bar{A}_i L_i^{\alpha} P_j} \right)^{1-\sigma} w_j L_j \tag{1}$$

$$\bar{W} = \frac{w_i}{P_i} \bar{u}_i L_i^{-\beta} \tag{2}$$

$$\sum_i L_i = \bar{L} \tag{3}$$

$$P_i = \left(\sum_k \left(\frac{\tau_{ki} w_k}{\bar{A}_k L_k^{\alpha}} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{4}$$

Equations and Unknowns

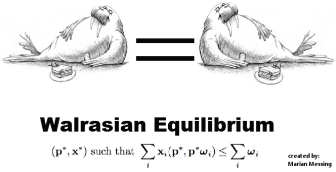
- Counting the number of equations in the system:
  - Equation (1) features N equations (one for each location $i = 1, 2, ..., N$)
  - Same for equations (2) and (4)
  - Equation (3) features only one equation.
  - Total: $3N + 1$

- Counting the number of unknowns (i.e. endogenous variables) in the system:
  - Variable w_i features N unknowns, one for each location: $w_{1,} w_{2,}, ..., w_N$
  - Same for variables P and L
  - Variable $\bar{W}$ features only one unknown (since welfare is the same everywhere)
  - Total: $3N + 1$

- Thus we have an equal number of equations and unknowns:
  - Suggests that the system has an unique solution for the set of endogenous variables $\{w_i, L_i, P_i, \bar{W}\}$

Walras's Law

- But wait! We actually have one fewer equation than it appears.
  - The feasibility equation (1) actually features $N-1$ equations, not N!
- Why? Walras's Law
  - A principle formulated by Leon Walras in 1874



- Important Corollary of the Law:
  - In an economy, if all markets but one are in equilibrium, then that last market must also be in equilibrium
  - For proof of Corollary, check Appendix slides

Walras's Law in the Spatial Model

- Apply Corollary to our spatial model's feasibility condition (equation (1)):
  - If $N-1$ markets are in equilibrium, then the Nth market must also be in equilibrium
  - Thus, if equation (1) holds for $i = 1, 2, ..., N-1$, then it must also hold for $i = N$
  - Nth equation doesn't add any new information!

- We thought we had N feasibility equations, but we actually have only $N-1$
  - The Nth equation is redundant!

- Thus, our equilibrium system only really has $3N$ equations, not $3N+1$
  - Which means we have fewer equations ($3N$) than unknowns ($3N+1$)
  - System of equations has a potentially infinite number of solutions!

Closing the Model: Price Normalization

- Therefore, we need one extra equation to "pin down" a single solution from the potentially infinite set of solutions.
- That's where price normalization comes in
- Pick a monetary variable and impose an arbitrarily chosen value to it:
  - For example, impose $w_1 = 1$ (a practical choice)
  - Note that other alternative choices would also be valid, e.g. $w_{32} = 12.93\sqrt{37}$ (less practical choice but still valid)
- With this extra price-normalization equation, we are back to $3N + 1$ equations
  - The equation system should now have a single solution.
  - We may now be able to find that solution, either analytically or using numerical methods.

Roadmap

- Normalization
- **Log-Linearization**

Log-Linearization

- In Lecture 13-14 we derived the following equation:

$$\tilde{X}_i = W^{1-\sigma} \sum_j (\tau_{ij}/\bar{A}_i)^{1-\sigma} \tilde{X}_j \tag{5}$$

- Applying logs and totally differentiating, we will arrive at the following equation:

$$d \ln \tilde{X}_i - \sum_j y_{ij} d \ln \tilde{X}_j - (1-\sigma) d \ln W = (1-\sigma) \sum_j y_{ij} d \ln(\tau_{ij}/\bar{A}_i) \tag{6}$$

- This process is called log-linearization
  - Equation system becomes expressed in terms of "changes" (hence the d operators) rather than levels

- Next, we show how to go from equation (5) to equation (6), step by step.

A Few Algebraic Facts

- Before we start, let's remember a few algebraic facts about logarithms:

$$\textbf{(F1)}: \ln(ab) = \ln(a) + \ln(b)$$

$$\textbf{(F2)}: \frac{d\ln(a)}{da} = \frac{1}{a}$$

$$\textbf{(F3)}: da = a \times d\ln(a)$$

$$\textbf{(F4)}: d\ln(a) = \frac{da}{a}$$

$$\textbf{(F5)}: d(m+n) = dm + dn$$

$$\textbf{(F6)}: \ln a^b = b \times \ln(a)$$

- Comments:
  - Note that (F3) and (F4) are consequences of (F2)
  - (F5) is reminiscent of the fact that the derivative is a linear operator: $\frac{d(f(x)+g(x))}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$

Log-Linearization (1)

- Let's start the log-linearization process. First take logs of equation (5) and use facts (F1) and (F6):

$$\ln \tilde{X}_i = (1-\sigma)\ln W + \ln\left(\sum_j (\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j\right)$$

- Now take differentials of both sides and use (F5):

$$d\ln \tilde{X}_i = (1-\sigma)d\ln W + d\ln\left(\sum_j (\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j\right)$$

- Apply (F4) to the latter term:

$$d\ln \tilde{X}_i = (1-\sigma)d\ln W + \frac{d\left(\sum_j (\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j\right)}{\sum_{j'} (\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}}$$

- Apply (F5) to bring the differential operator into the parentheses:

$$d\ln \tilde{X}_i = (1-\sigma)d\ln W + \frac{\sum_j d\left[(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j\right]}{\sum_{j'} (\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}}$$

Log-Linearization (2)

- Then use (F3) to transform the term inside the square brackets:

$$d\ln\tilde{X}_i = (1-\sigma)d\ln W + \frac{\sum_j \left[(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j\right] \times d\ln\left[(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j\right]}{\sum_{j'}(\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}}$$

- Now apply (F1), (F6), and (F5) in succession to term $d\ln\left[(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j\right]$ to obtain:

$$d\ln\tilde{X}_i = (1-\sigma)d\ln W + \frac{\sum_j \left[(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j\right] \times \left[(1-\sigma)d\ln(\tau_{ij}/\bar{A}_i) + d\ln\tilde{X}_j\right]}{\sum_{j'}(\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}}$$

Log-Linearization (3)

- Note that the denominator of the "big fraction", namely $\sum_{j'}(\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}$, doesn't depend on j.
  - Let us then bring it inside the "$\sum_j$" term:

$$d\ln\tilde{X}_i = (1-\sigma)d\ln W + \sum_j \left(\frac{(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j}{\sum_{j'}(\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}}\right)\times\left[(1-\sigma)d\ln(\tau_{ij}/\bar{A}_i)+d\ln\tilde{X}_j\right]$$

- Let's now define term $y_{ij} \equiv \frac{(\tau_{ij}/\bar{A}_i)^{1-\sigma}\tilde{X}_j}{\sum_{j'}(\tau_{ij'}/\bar{A}_i)^{1-\sigma}\tilde{X}_{j'}}$. This greatly simplifies notation:

$$d\ln\tilde{X}_i = (1-\sigma)d\ln W + \sum_j y_{ij}[(1-\sigma)d\ln(\tau_{ij}/\bar{A}_i) + d\ln\tilde{X}_j]$$

Log-Linearization (4)

- Distributing y_{ij} across the terms inside the square brackets and breaking up the summation, we get:

$$d \ln \tilde{X}_i = (1-\sigma) d \ln W + \sum_j y_{ij} (1-\sigma) d \ln(\tau_{ij}/\bar{A}_i) + \sum_j y_{ij} d \ln \tilde{X}_j$$

.

- Now we just have to rearrange terms across the two sides of the equation to get:

$$d \ln \tilde{X}_i - \sum_j y_{ij} d \ln \tilde{X}_j - (1-\sigma) d \ln W = (1-\sigma) \sum_j y_{ij} d \ln(\tau_{ij}/\bar{A}_i)$$

- This is the same as equation (6)
  - We've achieved our log-linearization goal!

Roadmap

- Normalization
- Log-Linearization
- **Appendix**

Appendix

- In this Appendix, we prove the Corollary of Walras's Law
- Namely, we prove that if all markets but one are in equilibrium, then that last market must also be in equilibrium
- Formal statement:
  - If equation (1) holds for $i = 1, 2, ..., N - 1$, then it must also hold for $i = N$

Proving the Corollary (1)

- For conciseness of notation, define the variable $\lambda_{ij} \equiv \left(\frac{\tau_{ij} w_i}{\bar{A}_i L_i^{\alpha} P_j}\right)^{1-\sigma}$
- For a given destination j, summing across all sources $i = 1, 2, ..., N$ yields:

$$\sum_{i=1}^{N} \lambda_{ij} = \sum_{i=1}^{N} \left(\frac{\tau_{ij} w_i}{\bar{A}_i L_i^{\alpha} P_j}\right)^{1-\sigma} = P_j^{\sigma-1} \sum_{i=1}^{N} \left(\frac{\tau_{ij} w_i}{\bar{A}_i L_i^{\alpha}}\right)^{1-\sigma} = P_j^{\sigma-1} P_j^{1-\sigma} = 1$$

  where the penultimate equality comes from the formula of the price index: $P_j^{1-\sigma} = \sum_j \left(\frac{\tau_{ij} w_i}{\bar{A}_i L_i^{\alpha}}\right)^{1-\sigma}$

- Multiply both sides of the equation by $w_j L_j$:

$$w_j L_j \sum_{i=1}^{N} \lambda_{ij} = w_j L_j \Rightarrow \sum_{i=1}^{N} \lambda_{ij} w_j L_j = w_j L_j$$

Proving the Corollary (2)

- Now, for both sides of the equation, take the summation across all j's:

$$\sum_{j=1}^{N}\sum_{i=1}^{N}\lambda_{ij}w_jL_j = \sum_{j=1}^{N}w_jL_j$$

- On the left-hand side, invert the order of the summations:

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}w_jL_j = \sum_{j=1}^{N}w_jL_j \tag{7}$$

- By hypothesis, we know the following equation holds for $i = 1, 2, ..., N-1$:

$$w_iL_i = \sum_{j}\lambda_{ij}w_jL_j$$

- So it seems the term $\sum_{j=1}^{N}\lambda_{ij}w_jL_j$ in equation (7) can be substituted by w_iL_i
  - But be careful! That only works for i from 1 to $N-1$

Proving the Corollary (3)

- So, before we can use the hypothesis, let's break "$\sum_{i=1}^{N}$" into two parts
  - One part is from $i = 1$ to $i = N - 1$. The other part is just the final term $i = N$
  - Equation (7) then becomes:

$$\left(\sum_{i=1}^{N-1} \sum_{j=1}^{N} \lambda_{ij} w_j L_j \right) + \sum_{j=1}^{N} \lambda_{Nj} w_j L_j = \sum_{j=1}^{N} w_j L_j$$

- Now we can apply the hypothesis to substitue $w_i L_i$ for $\sum_{j=1}^{N} \lambda_{ij} w_j L_j$ within the first summation:

$$\left(\sum_{i=1}^{N-1} w_i L_i \right) + \sum_{j=1}^{N} \lambda_{Nj} w_j L_j = \sum_{j=1}^{N} w_j L_j$$

- Rearranging terms:

$$\sum_{j=1}^{N} \lambda_{Nj} w_j L_j = \sum_{j=1}^{N} w_j L_j - \sum_{i=1}^{N-1} w_i L_i$$

Proving the Corollary (4)

- Now let's open up the terms in the right-hand side:

$$\sum_{j=1}^{N} \lambda_{Nj} w_j L_j = (w_1 L_1 + w_2 L_2 + ... + w_{N-1} L_{N-1} + w_N L_N)$$

$$-(w_1 L_1 + w_2 L_2 + ... + w_{N-1} L_{N-1})$$

- Cancelling terms on the right-hand side, we finally get:

$$\sum_{j=1}^{N} \lambda_{Nj} w_j L_j = w_N L_N$$

- That is precisely the feasibility condition (equation (1)) applied to the Nth location
  - That is, we have proved that the Nth market is also in equilibrium!
  - That concludes our proof